Prudence and Precaution for Natural Resource and Climate Uncertainty

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Chapter 1

Introduction

1.1 Background

1.1.1 Prudence and precautionary saving

Prudence from the Latin *prudentia* is defined in the Oxford English Dictionary as the quality of "acting with or showing care and thought for the future." It is one of the four cardinal virtues that Plato (427-347BC) describes in his 4th book of The Republic. The ancient Greeks and, later, Christian philosophers, most notably Thomas Aquinas (1225-1274), considered prudence as the cause, measure or even 'mother' of all virtues.[1] In economics, prudence is the channel through which uncertainty about future income leads to additional saving and postponing of consumption. This saving is precautionary in nature, as it acts to provide a buffer in case of an unexpected fall in future in income. An agent who is prudent now acts to set aside some of his current and known income for the future, which is inherently uncertain.

To illustrate the effect uncertainty can have on saving, Sandmo (1970) distinguishes two different types of uncertainty: uncertainty concerning future non-capital income and uncertainty concerning capital income. In terms of saving, the optimal response to these two types of uncertainty is opposite. Sandmo (1970) illustrates the first type of uncertainty (non-capital or labour income uncertainty) by citing Boulding (1966) [p. 535]:

> *"Other things being equal, we should expect a man with a safe job to save less than a man with an uncertain job."*

The second type of uncertainty (capital income uncertainty) is illustrated by citing Marshall (1920) [p. 226]:

[1] In contemporary economic policy debate, the term prudence often refers to the case in which less money is spent, for example in the austerity debate, in which it often juxtaposed with profligacy (e.g. "Prudence and profligacy," The Economist, 12th September 2015).

> *"The thriftlessness of early times was in great measure due to the want of security that those who made provision for the future would enjoy it: only those who were already wealthy were strong enough to hold what they had saved; the laborious and self-denying peasant who had heaped up a little store of wealth only to see it taken from him by a stronger hand, was a constant warning to his neighbours to enjoy their pleasure and their rest when they could."*

These two quotations illustrate, respectively, prudence and risk aversion. The prudent agent in the first quote cannot change the level of uncertainty he is exposed to, but can change how much hardship or dis-utility the uncertainty brings by saving to create a buffer. Assuming the nature of employment is given, the agent who has the uncertain job acts prudently and saves. With greater savings, the uncertainty is easier to bear. The risk-averse agent in the second quotation, on the other hand, can control the level of uncertainty he is exposed to. By saving less, he avoids the uncertainty that goes hand in hand with a store of wealth exposed to the risk of expropriation. Clearly, this has a cost; the agent is less able to optimally smooth the proceeds of his labour over time. The agent in the second quotation may be both prudent and risk averse, but it is his risk aversion that dominates his behaviour.

It has been known since the work of the mathematician and fluid dynamicist Daniel Bernoulli (1700-1782) that risk-aversion is associated with the concavity of the utility function.[2] Pratt (1964) and Arrow (1965) introduced $CARA = -U''(C)/U'(C)$ and $CRRA = -CU''(C)/U'(C)$ as measures of the strength of risk aversion, known respectively as the coefficient of absolute risk aversion ($CARA$) and the coefficient of relative risk aversion ($CRRA$), with $U(C)$ denoting the von Neumann-Morgenstern utility function of consumption C. The first to describe prudence in a formal framework, Leland (1968) and Sandmo (1970) have shown that prudence is associated with a positive third derivative of the utility function.[3]

The mechanism behind prudence is most easily illustrated by considering the effect of a small amount of additional income ΔI on the level of risk aversion, as described the (negatively-signed) second derivative of the utility function with respect to consumption $-U''(C)$. Using the leading-order term in a Taylor-series expansion about the initial level of consumption C_0, this effect can be approximated by:

$$-U''(C_0 + \Delta I) \approx -U''(C_0) - U'''(C_0)\Delta I. \tag{1.1.1}$$

If the third derivative of the utility function is positive ($U'''(C_0) > 0$), the decision maker is said to be prudent. Additional income ΔI will then act to decrease the level of risk aversion ($-U''(C_0 + \Delta I)$ in (1.1.1)). In words, a decision maker who has

[2]Bernoulli (1738) translated by Sommer (1954).

[3]Their two-period results were extended to a multi-period setting by Sibley (1975) and Miller (1976).

more income is less sensitive to risk.[4] Kimball (1990) has shown that the theory of precautionary saving is, in fact, isomorphic to the Arrow-Pratt theory of risk aversion resulting in analogous definitions for the coefficients of absolute (*CAP*) and relative prudence (*CRP*):

$$CAP = -\frac{U'''(C)}{U''(C)} \quad \text{and} \quad CRP = -\frac{CU'''(C)}{U''(C)}. \tag{1.1.2a,b}$$

Provided utility is additively separable, the coefficient of absolute prudence *CAP* is the appropriate measure of the strength of the precautionary savings motive and, if *CAP* is an increasing function of consumption (for example when *CRP* is constant), then labour income uncertainty will lower the marginal propensity to consume out of wealth at any level of consumption (Kimball 1990). For a simple iso-elastic utility function of the form $U(C) = C^{1-\eta}/(1-\eta)$, one has a clear relationship between the coefficients of relative risk aversion and relative prudence: $CARA = \eta$ and $CRP = \eta + 1$.

When prudence leads to precautionary saving

Noting that closed-form solutions for consumption with stochastic labour income had not been derived, Zeldes (1989) was the first to employ numerical methods to approximate the solution for optimal consumption. He emphasized the potentially dramatic difference between the resulting consumption function compared to the certainty equivalent solution. Provided the consumer is prudent,[5] optimal initial consumption is a concave function of initial non-financial wealth, known as the consumption function. Correspondingly, the marginal propensity to consume is a decreasing function of financial wealth: poor consumers have a greater marginal propensity to consume out of a windfall than rich consumers. When non-stochastic financial assets are small relative to uncertain human capital, the level of precautionary saving is high.

Carroll and Kimball (1996)[6] were the first to provide an analytical explanation for the old idea that the consumption function is concave, an idea which they date back to Keynes (1936). They showed that for a class of utility functions that display hyperbolic absolute risk aversion (HARA), the consumption function is concave provided the following inequality holds: $k \equiv U'''(C)U'(C)/(U''(C))^2 > 0$, noting that constant absolute risk-aversion (CARA) utility functions satisfy $k = 1$ and constant relative risk-aversion (CRRA) utility functions $k > 1$.[7] Furthermore,

[4]In an alternative but analogous definition, a decision maker is prudent when displaying greater aversion to negative than positive shocks to income. In (1.1.1), a negative shock ($\Delta I < 0$) clearly reduces the degree of risk aversion for $U'''(C_0) > 0$, whereas a positive shock has the opposite effect. In this alternative interpretation ΔI is no longer transferable income, but an income shock.

[5]Zeldes (1989) has $CRP = 3$ and constant relative risk-aversion utility.

[6]Relying on work by Neave (1971) and Sibley (1975).

[7]Huggett and Vidon (2002) emphasize that the commonly quoted criterion that the third derivative of the utility function must be positive is thus not a sufficient condition for precautionary

Carroll and Kimball (1996) show that this concavity is strict except for two cases: CARA utility with uncertain labour income (with a deterministic rate of return on assets) and CRRA utility with an uncertain rate of return on assets (without uncertain labour income).[8] In these cases, the consumption function is linear.[9]

Prudence and precautionary saving in the economy at large

From an empirical perspective, it is often conjectured that a significant share of the world's savings can be explained by uninsured income and corresponding precautionary saving. This conjecture is supported by both household surveys, in which 'emergencies' are reported as reasons for saving, as well as by more structural empirical studies (see Browning and Lusardi (1996) and Carroll and Kimball (2008) for reviews). In their review, Carroll and Kimball (2008) distinguish four categories of empirical evidence: Euler equation methods, structural estimation using micro data, regression evidence and survey evidence, which are summarized below following the discussion in Carroll and Kimball (2008).

The oldest empirical literature dating from Hall (1978) sets out to empirically test the consumption Euler equation and derive values for coefficient of risk aversion from household data. Carroll (2001),[10] reviewing earlier work, shows that such methods, based on approximations to the Euler equation, suffer from edogeneity problems that are ultimately related to the fact that, in expectation, consumption and income have to grow at the same rate in the long term, even in the presence of uncertainty. Using a structural model of optimal life-cycle consumption with labour income uncertainty and Consumer Expenditure Survey data, Gourinchas and Parker (2002) aim to overcome such endogeneity problems. They decompose saving into precautionary and life-cycle components and find coefficients of relative risk aversion varying between 0.5 and 1.4, estimates which have been improved by later studies by taking into account job mobility and better specification of uncertainty (e.g. Low et al. (2010)). Abstracting from a structural model, other authors (e.g. Carroll and Samwick (1997) and Lusardi (1998)) have tackled the problem by directly regressing (precautionary) wealth on its possible explanatory variables. Carroll and Samwick (1997), for example, find that 20-50% of U.S. household saving is precautionary. As a structural model is absent, it is not

wealth accumulation, but the above criterion by Carroll and Kimball (1996) needs to be satisfied. Huggett and Ospina (2001) show that in the presence of liquidity constraints precautionary saving may arise even without a positive third derivative.

[8] The latter corresponds to the well-known capital asset pricing model (CAPM) solutions of Merton (1971), which I return to in Chapter 4.

[9] The two categories of uncertainty here correspond to the two categories of uncertainty in Sandmo's (1970) example presented above: uncertainty concerning future non-capital income (labour income) and uncertainty concerning the yield on capital income (asset returns). Although only uncertainty of the first type will play a role in Chapters 2 and 3, both types will feature in Chapters 4 and 5.

[10] Endearingly entitled "Death to the log-linearized consumption Euler equation! (and very poor health to the second-order approximation)".

possible to translate such findings into estimates of structural parameters such as the coefficient of risk aversion and the coefficient of prudence. Finally, consumers may be directly asked for their target level of precautionary wealth. Based on the 1995 and 1998 U.S. Survey of Consumer Finance, Kennickell and Lusardi (2004) find that consumers, on average, report a target level of precautionary wealth of 8% of total net worth and 20% of financial wealth. Using a carefully designed survey to deduce underlying preference parameters, Kimball et al. (2008) estimate a coefficient of relative risk aversion with a median of 6.3 and a mean of 8.2.

Gollier (2012, 2018) provide a powerful overview of the normative implications of prudence and precaution for optimal discounting. When inverted, the stochastic Keynes-Ramsey rule describes the discount rate $r^\star$ that should be used in the valuation of an incremental, deterministic project in an economy facing uncertain economic growth prospects. In the most simple case of an economy whose growth rate is described by a Brownian motion with mean g and volatility σ and for a CRRA utility function of the form $U(C) = C^{1-\eta}/(1-\eta)$, the optimal discount rate $r^\star$ is given by (e.g. Gollier (2012), chapter 3)

$$r^\star = \underbrace{\rho}_{\substack{\text{time}\\ \text{impatience}}} + \underbrace{\eta g}_{\substack{\text{economic}\\ \text{growth}}} \underbrace{- \frac{1}{2}(1+\eta)\eta\sigma^2}_{\text{prudence}}, \tag{1.1.3}$$

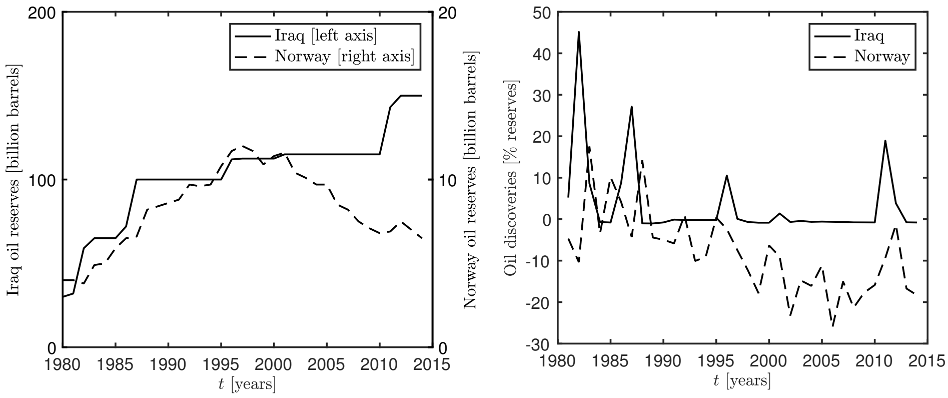
where the social or utility discount rate ρ captures time impatience, and η plays both the role of coefficient of intergenerational inequality aversion (or the inverse of the elasticity of intertemporal substitution) and of the coefficient of relative risk aversion ($CRRA = \eta$).[11] Deterministically, the optimal discount rate $r^\star$ is given by the sum of the social discount rate ρ and the product of η, in its role of coefficient of intergenerational inequality aversion, and the growth rate g. More impatience and wealthier future generations result in future costs and benefits being discounted more. Uncertainty in economic growth, provided the decision maker is prudent ($CRP = 1 + \eta > 0$, cf. (1.1.2)), acts in the opposite way. It reduces the optimal discount rate $r^\star$, thus increases the weight put on the costs and benefits of future generations and makes us more precautious.

This thesis focusses on the optimal, prudent response to two types of uncertainty that are very different to household income risk or economic growth uncertainty: natural resource uncertainty and climatic uncertainty, introduced below. The optimal saving, investment, mitigation or abatement response to these two

[11] As for all long-horizon problems in economics, the non-stochastic discount rate evidently also plays a decisive role in the problems studied. Furthermore, in a framework with time-separable utility, one parameter plays at least two roles: constant of relative risk aversion and coefficient of intergenerational inequality aversion or the inverse of the elasticity of intertemporal substitution. Epstein-Zin preferences constitute a specification of recursive utility that allows risk aversion and intertemporal substitution to be separated identifying two separate coefficients (Epstein and Zin (1989) and Duffie and Epstein (1992) for continuous time). I return to these issues in Chapters 4 and 5.

Figure 1.1: Historical oil reserves and discoveries for Iraq and Norway.

(a) Reserves (b) Discoveries



types of uncertainty lies outside of the realm of household behaviour, and observed behaviour thus offers only limited guidance for optimal policy. Natural resource and climatic uncertainty call for explicit government intervention. This thesis therefore takes a normative approach rooted in welfare economics: it argues how much should be saved and how much action should be taken based on the policy maker's subjective preference parameters, for which I take the above values as an indicative range. Assuming a constant relative risk-aversion utility function, the range of *CRRA* values reported above, between *CRRA* = 0.5 (Gourinchas and Parker 2002) and *CRRA* = 8.2 (Kimball et al. 2008), correspond to *CRP* = 1.5-9.2. In this thesis coefficients of relative prudence in the range 1.5-10 are used (see also the discussion in chapters 1-3 of Gollier (2012)).

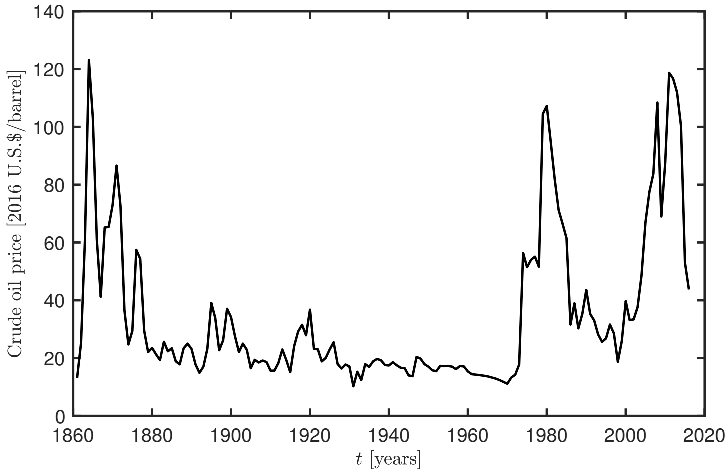
1.1.2 Natural resource uncertainty

The value of a country's natural resource base is subject to two major sources of uncertainty:[12] uncertainty regarding their quantity, as extractable reserves are never known with certainty and the prowess of extractive technologies is subject to shocks, and uncertainty regarding their price. Figure 1.1 illustrates the first type of uncertainty: uncertainty regarding reserves.[13] With reserves shown in panel a and new discoveries in panel b, it is clear that the reserve stock can be subject to large

[12]Extraction costs constitute a third type of uncertainty, which may be partly explained by the price of the resource (Toews and Naumov 2015).

[13]Data from BP (2015) showing so-called proved reserves, those quantities that geological and engineering information indicates with reasonable certainty can be recovered in the future from known reservoirs under existing conditions. As a result of the qualification "under existing conditions," which is commonly made in reserve estimation, shock to reserves are not necessarily independent from shocks to prices.

Figure 1.2: Historical crude oil price.



shocks. For the two countries that are considered in Chapter 2, Iraq and Norway, these shocks can be as large as 45% and 25% of the reserve stock, respectively. In fact, for many countries discovery rates have historically exceeded extraction rates, resulting in permanently increasing reserves (Gelb et al. 2012). Especially for resource-rich countries, fiscal prudence and the ultimate size of natural resource reserves are intimately related (Hamilton and Atkinson 2013).

Figure 1.2 illustrates the second type of uncertainty: price uncertainty.[14] Many authors have attempted to explain the crude oil price and the shocks driving it (Hamilton 2009, Kilian 2009, Anderson et al. 2014, Venables 2014). Hamilton (2009) concludes that crude oil prices have historically tended to be notoriously difficult to predict and that changes in the real price of oil have tended to be permanent and governed by very different regimes at different points in time. What is evident from Figure 1.2 is that variability is the dominant feature of the crude oil price, stretching almost an order of magnitude over a time scale of a decade. This picture is only reinforced by the recent slump in crude oil prices followed by the partial recovery not shown in Figure 1.2 with current (April 2018) prices at between 60 and 70 U.S.$/barrel. In this thesis, I will only consider the second type of uncertainty: price uncertainty.

1.1.3 Climatic uncertainty

One of the few certainties in the climate change debate is the highly uncertain nature of the degree and impact of future global warming. This uncertainty is in large

[14]Compiled by BP (2015) from US Average (1861-1944), Arabian Light posted at Ras Tanura (1945-1983) and Brent (1984-2014).

part due to the extremely long time scales over which today's emissions will impact global warming and cause economic damage, but also due to the complexity of the climate system. Understanding optimal policy requires the coupling of two dynamical systems: the physical climate system and the optimization equations of welfare economics. Four categories of uncertainty can be distinguished:[15] the uncertainty inherent in the carbon cycle,[16] the uncertain degree to which the atmospheric carbon stock generates warming, the uncertain impact of warming on production and, finally, the uncertain evolution of total factor productivity.[17] Although the four different types of uncertainty are evidently interrelated, I briefly discuss each type in turn.

Partly motivated by the ultimate need to model its volatility for pricing on financial markets, McAleer and Chan (2006) use data from the Mauno Lao data set to model trends and volatility in atmospheric carbon dioxide concentrations. Figure 1.3b shows the data these authors use with the dashed line denoting annual averages.[18] It is immediately evident that the seasons are the main source of variability and the annual averages over the period considered are very smooth.[19] Figure 1.3a shows much longer term data from the Law Dome Ice Core 2000-year data set.[20] It is evident from this panel that larger variability may be observed over longer time scales. An emerging literature has examined non-marginal calamities or tipping points in the dynamical climate system (e.g. Pindyck (2013b), Lemoine and Traeger (2014), van der Ploeg and de Zeeuw (2018)). Shocks to the atmospheric greenhouse gas concentration, caused by their sudden release into the atmosphere, may act as the cause of such abrupt changes, which may be irreversible (Alley et al. 2003). Examples include the thawing of permafrost soils and the consequent enhanced decomposition of organic matter resulting in release of significant quantities of carbon and methane (Dutta et al. 2006, Köhler et al. 2014) and abrupt changes in the thermohaline circulation, which may change the capacity of

[15] Ackerman and Stanton (2012) also identify four categories of "uncertainty": uncertain climate sensitivity, different damage function estimates at low temperatures and at high temperatures and choice of the discount rate. Lemoine (2017) effectively considers two types of uncertainty: uncertainty in consumption and uncertainty in the climate system captured through temperature. See Dijkstra (2013) for a review of sources of variability in the physical climate system and their different time scales.

[16] In this thesis carbon is generally used as a short hand for greenhouse gases and, given the stylized nature of the modelling, both terms are used interchangeably.

[17] I have implicitly assumed that the level of emissions and consumption are directly controlled by policy, through a carbon tax or otherwise. In other words, they are forward-looking variables and thus not direct sources of uncertainty. Instead, these variables respond optimally to uncertainty.

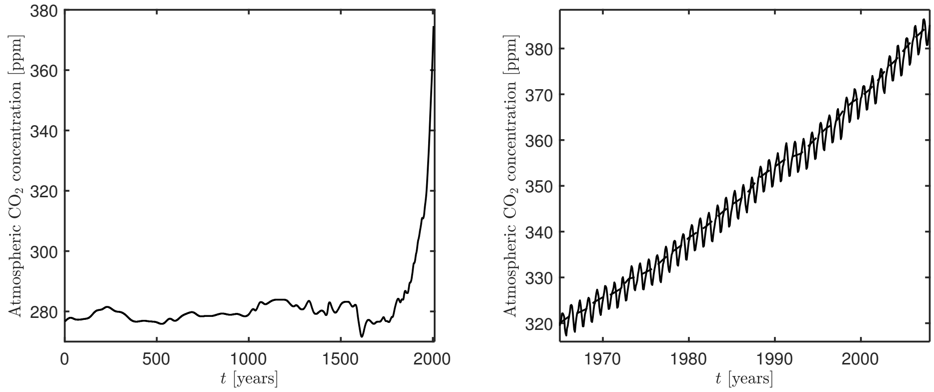
[18] Monthly data for January 1965 to December 2002 (shown here including more recent data up to December 2008) from in situ air samples collected at Mauna Loa, Hawaii, USA and available online at `http://cdiac.ornl.gov/ftp/trends/co2/maunaloa.co2`.

[19] McAleer and Chan (2006) estimate a mean of 344.5 ppm and a standard deviation of 16.0 ppm giving a normalized volatility of 0.05 (or 5%).

[20] Annual data from the Law Dome firn and ice core records and the Cape Grim record available online at `ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/law/law2006.txt`.

Figure 1.3: Historical atmospheric CO_2 concentrations.

(a) Long-term annual data (b) Short-term monthly data



the ocean to act as a carbon sink (Clark et al. 2002).

It is evident from Figure 1.4 that the historical temperature is subject to much greater variability than the atmospheric carbon stock.[21] Equilibrium climate sensitivity is the equilibrium change in global and annual mean surface air temperature after doubling the atmospheric concentration of CO_2 relative to pre-industrial levels. It is exactly this quantity that has proven notoriously difficult to predict and remains associated with a broad and significantly right-skewed probability distribution (IPCC 2013). Roe and Baker (2007) have argued that, due to the nature of the climate system and its built-in feedback loops, the breadth of the probability distribution of climate sensitivity is relatively insensitive to decreases in uncertainties associated with the underlying climate processes. According to the latest (AR5) IPCC report IPCC (2013), equilibrium climate sensitivity is "likely" in the range 1.5°C to 4.5°C ("high confidence"), "extremely unlikely" less than 1°C ("high confidence"), and "very unlikely" greater than 6°C ("medium confidence").[22] On a fundamental level, the properties of the climate system that can be observed now cannot be used to distinguish between a climate sensitivity of 4°C or >6°C, as the conditions at 4°C will be fundamentally different. Yet, despite this, finding an upper bound on the climate sensitivity has become the "holy grail of climate research" (Allen and Frame 2007). The inherent uncertainty associated with the upper tail of the distribution of climate sensitivity, motivates Allen and Frame (2007) to call off

[21] Annual data from the HadCRUT3 database available online at `http://www.cru.uea.ac.uk/cru/data/temperature/`.

[22] The lower temperature limit of the assessed likely range is less than the 2°C in AR4 (IPCC 2007), but the upper limit is the same. See also Newbold and Daigneault (2009) for a review of climate sensitivity estimates and the resulting probability distribution.

the quest for and upper limit to climate sensitivity.

Coupling the physical climate system with the economic welfare optimizing system, the effects of changes in temperature must be translated into economic losses through the specification of so-called damage functions. Most integrated assessment models, models which integrate the physical climate system with an economic model for the welfare costs of climate change, relate temperature increases and the economy's output through a loss function that is proportional to the gross domestic product in the absence of climate change and a (nonlinear) function of temperature. For example, Nordhaus (2008) assumes an inverse-quadratic loss function (of temperature) with two parameters and Weitzman (2009) an exponential-quadratic (of temperature) function with one parameter. All such loss functions are arbitrary, do not rely on economic theory and are calibrated to very little or in fact no data (Pindyck 2013a). As a result of this, as well as more fundamentally, they represent a considerable source of uncertainty.

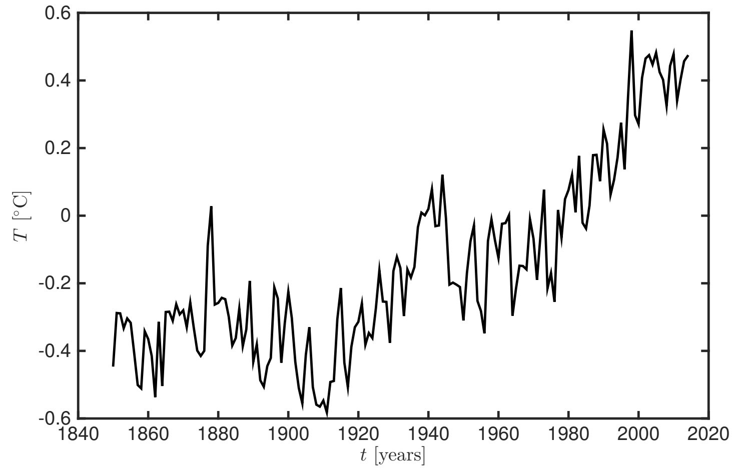
Bansal and Ochoa (2011a) use evidence from global capital markets to show that temperature is an aggregate risk factor that adversely affects economic growth. The covariance between country equity returns and temperature (i.e. temperature betas) contains sharp information about the cross-country risk premium; countries closer to the Equator carry a positive temperature risk premium, which decreases as one moves farther away from the Equator. Dell et al. (2012) examine the historical relationship between temperature fluctuations and economic growth and find substantial effects of temperature shocks, but also only in poor countries. In poor countries, a 1°C rise in temperature in a given year reduces economic growth by 1.3 percentage points on average (Dell et al. 2012). Crucially, temperature affects the growth rate and not the level of GDP. Finally, if the loss function is assumed to be proportional to GDP, non-climatic shocks to world GDP will also have an important role to play in the valuation of the social costs of emitting carbon.

1.1.4 Perturbation methods

Due to the role of higher-order derivatives, namely the third derivative of the utility function, closed-form solutions to economic models displaying prudence are generally unavailable, with two notable exceptions discussed in §1.1.1. Two categories of approximate methods are available in such cases: numerical methods and perturbation methods.[23] This thesis makes use of a continuous-time framework. As a result, the solution to the stochastic optimization problem is a (system of) partial differential equation(s). Apart from the ubiquitous (log-) linearization in linear-quadratic control, perturbation methods are relative uncommon in economics with

[23] See Judd (1998) and Miranda and Fackler (2002) for comprehensive overviews of the numerical methods commonly applied in economics and finance.

Figure 1.4: Historical global mean surface temperature anomaly with respect to the period 1961-1990.



the notable exception of mathematical finance.[24]

Perturbation theory comprises mathematical methods for finding an approximate solution to a problem by starting from the exact solution of a related, simpler problem. In the case of this thesis, the simpler problem is sometimes the deterministic problem, which is perturbed by introducing uncertainty (Chapter 2), or the stochastic endogeneous growth model of the AK-type, which is perturbed by introducing climate change (Chapter 5). The advantage of using perturbation methods instead of numerical methods is twofold: the channel through which uncertainty modifies the deterministic solution can be readily identified and interpreted, and the method allows for leading-order estimates that can be evaluated without undue attention to numerical convergence and stability. In other words, perturbation methods are simple and reliable. The general approach taken in this thesis is illustrated below using two examples that play a role throughout the thesis and were introduced in §1.1.1: an example with capital income uncertainty and an example with non-capital income uncertainty.

[24]See Hinch (1991) for an introduction to perturbation or asymptotic methods, Bender and Orszag (1999) for an overview of their use in the natural sciences and engineering and Fouque et al. (2011) for a review of the use of the so-called separation of scales or multiple-scales method, a class of perturbation methods, in mathematical finance. The separation of scales method is used in Chapter 5. There are a few authors in economics, who make use of higher-order perturbations going beyond linear-quadratic control (e.g. Schmitt-Grohé and Uribe (2004), Benigno et al. (2013)), but these authors formulate their models in discrete time.

Example with capital income uncertainty

It is instructive to examine how perturbation methods may be applied to a well-known problem, for which a closed-form solution is available: the classical capital asset pricing model (CAPM) of Merton (1971). Consider a decision maker who has access to one risky and one risk-free asset to transfer income to the future. A share w is allocated to the risky asset, which is modelled by a geometric Brownian motion process with drift α and volatility σ. The remainder $1-w$ is allocated to the risk-free asset with return r ($\alpha > r$). In the absence of other income, the decision maker's Hamilton Jacobi Bellman equation is given by:

$$\rho J(W) = \max_{w,C}\Big[U(C) + \frac{1}{dt}\mathrm{E}_t[dJ(W)]\Big], \tag{1.1.4}$$

where $J(W)$ is the value function, consumption C and the risky-asset weight w are chosen optimally, and the decision maker discounts utility U at rate ρ. Total wealth W is accumulated according to:

$$dW = w(\alpha - r)Wdt + rWdt - Cdt + wWdZ, \tag{1.1.5}$$

where $dZ \sim N(0, dt)$ is a Wiener process. Itō's differential operator applied to the value function in (1.1.4) gives:

$$\frac{1}{dt}\mathrm{E}_t[dJ(W)] = J_W\Big(w(\alpha - r)W + rW - C\Big) + \frac{1}{2}J_{WW}w^2W^2\sigma^2, \tag{1.1.6}$$

where subscripts are used to denote partial derivatives (e.g. $J_W \equiv \partial J/\partial W$). By taking first-order conditions with respect to w and C and differentiating (1.1.4) with respect to W invoking the envelope condition, two simultaneous differential equations can be obtained:

$$w = \frac{\alpha - r}{\sigma^2\eta}\frac{C/W}{C_W}, \tag{1.1.7}$$

$$\frac{1}{dt}\mathrm{E}_t[dC] = \frac{1}{\eta}(r-\rho)C + \frac{1}{2}CRP\left(\frac{\partial C}{\partial W}\right)^2\frac{w^2W^2\sigma^2}{C^2}C, \tag{1.1.8}$$

where a CRRA utility function of the form $U(C) = C^{1-\eta}/(1-\eta)$ has been adopted with $CRP = 1 + \eta$. Combining (1.1.7) and (1.1.8), a single-variable equation in consumption C can be obtained:

$$\frac{1}{dt}\mathrm{E}_t[dC] = \frac{1}{\eta}(r-\rho)C + \frac{1}{2}(1+\eta)\frac{(\alpha - r)^2}{\sigma^2\eta^2}C, \tag{1.1.9}$$

To avoid singular perturbations,[25] the volatility and the excess return are assumed to scale equivalently: $\sigma^2 = O(\epsilon)$ and $(\alpha - r) = O(\epsilon)$, where ϵ is the small parameter.

[25] In the limit of $\sigma \to 0$ with $(\alpha - r)$ finite, the asset allocation problem is of course ill-defined. One asset would then have a greater return than the other ($\alpha > r$), and an infinite amount of the former would by purchased by borrowing an infinite quantity of the latter.

The differential equations (1.1.7) and (1.1.9) can now be solved by introducing asymptotic series solutions of the form:

$$C = C^{(0)} + C^{(1)}\epsilon + C^{(2)}\epsilon^2 + ... \quad \text{and} \quad w = w^{(0)} + w^{(1)}\epsilon + w^{(2)}\epsilon^2 + ... \tag{1.1.10a,b}$$

The deterministic solution corresponds to $C^{(0)}$ and $w^{(0)}$. At zeroth order, applying Itō's differential operator, (1.1.9) gives:

$$C_W^{(0)}\Big(rW - C^{(0)}\Big) = \frac{1}{\eta}(r-\rho)C^{(0)}, \tag{1.1.11}$$

which has the following solution for $C^{(0)}$ and the corresponding solution for $w^{(0)}$ from (1.1.7):

$$C^{(0)} = \Big(r - \frac{1}{\eta}(r-\rho)\Big)W \quad \text{and} \quad w^{(0)} = \frac{\alpha - r}{\eta\sigma^2}. \tag{1.1.12a,b}$$

Using the zeroth-order solutions (1.1.12), (1.1.9) gives at the next order:

$$\begin{aligned} \frac{C_W}{C}\Big(\underbrace{w^{(0)}(\alpha - r)W}_{O(\epsilon)} + \underbrace{rW}_{O(1)} - C\Big) + \frac{C_{WW}}{2C}\underbrace{(w^{(0)})^2 W^2\sigma^2}_{O(\epsilon)} &= \\ \underbrace{\frac{1}{\eta}(r-\rho)}_{O(1)} + \underbrace{\frac{1}{2}(1+\eta)\frac{(\alpha-r)^2}{\sigma^2\eta^2}}_{O(\epsilon)}&, \end{aligned} \tag{1.1.13}$$

where C contains terms up to and including $C^{(1)}$. Ignoring higher-order terms, (1.1.13) has the solution:

$$C = C^{(0)} + C^{(1)}\epsilon = \Big(r - \frac{1}{\eta}(r-\rho) - \frac{1}{2}(1-\eta)\frac{(\alpha-r)^2}{\sigma^2\eta^2}\Big)W, \tag{1.1.14}$$

Finally, from (1.1.7) $w = w^{(0)} + w^{(1)}\epsilon = w^{(0)} = (\alpha - r)/(\eta\sigma^2)$ and $w^{(1)} = 0$. In fact, it can be shown that all higher-order contributions to w and C are zero and that the exact solutions to (1.1.4) by Merton (1971) correspond to (1.1.12b) and (1.1.14). The leading-order non-linear terms are sufficient to recover the full solution in this particular case. In general, a larger or an infinite number of terms[26] is required to exactly solve the relevant equations. This thesis exclusively focusses on the leading-order contribution uncertainty makes, which is exact in the above example, but approximate in the following example.

Example with non-capital income uncertainty

Consider an intertemporally optimizing decision maker with permanent and constant deterministic income $I_D \neq I_D(t)$ and autonomous, temporary, declining and stochastic income evolving according to (cf. a resource windfall):

$$dI_S \sim N(-kdt, I_S^2\sigma^2 dt) \quad \text{for} \quad t \leq T, \tag{1.1.15}$$

[26]Hence the term asymptotic.

which represents a linearly declining stream of income for duration $T = I_S/k$ and which is subject to shocks that are proportional to its magnitude. The decision maker has access to a single risk-free asset:

$$\frac{dB}{dt} = rB + I - C, \tag{1.1.16}$$

where $I = I_D + I_S$. The decision maker discount utility at rate ρ and has the corresponding Hamilton Jacobi Bellman equation:

$$\rho J(t, I, B) = \max_{w,C} \Big[U(C) + \frac{1}{dt}\mathrm{E}_t[dJ(t, B, I)]\Big], \tag{1.1.17}$$

with Itō's differential operator applied to the value function giving:

$$\frac{1}{dt}\mathrm{E}_t[dJ(t, B, I)] = J_t + J_B(rB + I - C) - J_I k + \frac{1}{2}J_{II}I_S^2\sigma^2. \tag{1.1.18}$$

Taking the first-order condition with respect to C and differentiating (1.1.17) with respect to B invoking the envelope condition gives an Euler equation of familiar form:

$$\frac{1}{dt}\mathrm{E}_t[dC] = \frac{1}{\eta}(r - \rho)C + \frac{1}{2}CRP\frac{\left(\frac{\partial C}{\partial I_S}\right)^2 I_S^2\sigma^2}{C^2}C. \tag{1.1.19}$$

As in the previous example, proceed by defining a series solution: $C = C^{(0)} + C^{(1)}\epsilon + C^{(2)}\epsilon^2 + ...$ with $\sigma^2 = O(\epsilon)$ and considering subsequent terms in turn. At zeroth order and after some manipulation, optimal consumption is a fixed share of the net present value of all assets:

$$C^{(0)}(t) = \left(r - \frac{r - \rho}{\eta}\right)\left(B(t) + \frac{rI_S(t) - (1 - \exp(-I_S r/k))}{r^2} + \frac{I_D}{r}\right) \text{ for } t \le \frac{I_S(t)}{k}, \tag{1.1.20}$$

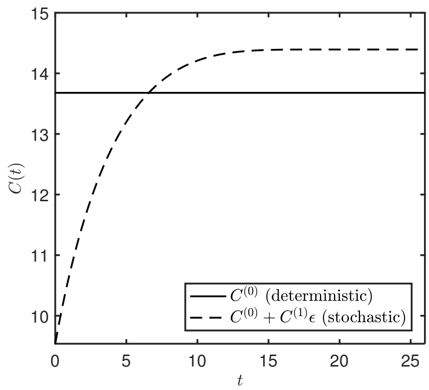
where the second term in the right-hand side brackets simply represents the net present value of the expected linearly declining income stream I_S discounted at rate r. The deterministic solution (1.1.20) can be readily used to approximate the partial derivative in (1.1.19) at the next order of approximation:

$$\frac{1}{C}\frac{1}{dt}\mathrm{E}_t[dC] = \frac{1}{2}CRP\left(\underbrace{1 - \exp(-I_S r/k)}_{\substack{\text{marginal propensity}\\ \text{to consume}}}\right)^2 \underbrace{\frac{I_S^2\sigma^2}{C^2}}_{\substack{\text{relative size}\\ \text{of shocks}}} \tag{1.1.21}$$

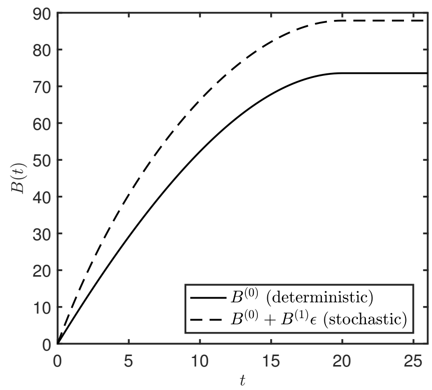
where I have set $\rho = r$ for convenience, and C contains terms up to and including $C^{(1)}$. The prudential tilt of the consumption path in (1.1.21) is locally proportional to the coefficient of relative prudence, the (square of the) marginal propensity to consume out of shocks $\partial C/\partial I_S$ and the (square of the) size of the shocks as a fraction of total consumption. The marginal propensity to consume, in turn, depends

Figure 1.5: Perturbation solutions for non-capital income uncertainty.

(a) Consumption



(b) Assets



Parameters and coefficients used: $\rho = r = 0.05$, $\eta = 9$ ($CRP = 10$), $I_D = 10$, $I_S(0) = 10$, $k = 0.5$ ($T = 20$) and $\sigma = 0.25$.

on the duration of the windfall $I_S r/k = rT$, with longer windfalls necessitating greater amounts of precautionary saving.

Equation (1.1.21) and the simultaneous equations for $(1/dt)\mathrm{E}_t[dI_S]$ (1.1.15) and dB/dt (1.1.16) can now be solved as an initial value problem subject to the terminal condition represented by the deterministic stable manifold at $t = T$ ($I_S = 0$). The perturbative approach has reduced the system from coupled partial differential equations to coupled ordinary differential equations, which can be solved using standard Runge-Kutta techniques. Figure 1.5 compares the zeroth-order or deterministic solution and the first-order or stochastic solution for consumption and asset build-up. It is evident that prudence and uncertainty lead to less consumption initially and the build-up of more assets: precautionary saving.

1.2 Synopsis and layout

This thesis examines the optimal policy response to the uncertainty associated with natural resource revenues and the uncertainty associated with climate change. Starting from a continuous-time, time-separable, dynamic stochastic welfare optimisation framework, this thesis uses perturbation methods to develop leading-order estimates of the effect of uncertainty in combination with risk aversion and prudence. Its objectives are two-fold: improve our understanding of the mechanisms through which uncertainty acts and provide order-of-magnitude estimates of the effects and thus be able to assess their importance.

Regarding natural resource uncertainty, this thesis examines three related topics. First, this thesis firstly aims to address the question how resource-rich countries should allocate the temporary and highly volatile income they receive from extracting natural resources. This income, sometimes referred to as windfall income because of its temporary nature, can be either consumed, invested or saved. A rationale for saving may arise out of concerns for intergenerational equality and, as such, is simply a manifestation of the permanent income hypothesis, or because the policy maker is prudent, in which case the saving is precautionary in nature. Although the focus in this thesis is on the precautionary component of saving, such saving can never be considered in isolation and alternatives are therefore considered in this thesis. For developing economies, where public capital is scarce and debt burdens are high, paying off sovereign debt and investing in public capital may be more important driving forces.

Second, this thesis considers the optimal portfolio allocation of the above-ground saving fund, in which the proceeds from extracting the below-ground resource are parked. An optimally chosen financial portfolio will reduce the aggregate level of uncertainty to which the economy is exposed, by choosing assets that offset oil price risk. If such a portfolio is unavailable, additional precautionary saving may be required. Third, this thesis examines how the resource extraction decision may be used to optimally control the stream of resource revenue.

Regarding climate uncertainty, this thesis examines the effect of uncertainty on estimates of the social cost of carbon and thence the optimal carbon tax. This is achieved in the context of a stylized integrated assessment model based on an endogenous growth model. A combination of different perturbation methods is used to develop simplified rules for the social cost of carbon and its dependence on the four categories of uncertainty identified in §1.1.3: shocks to the carbon cycle, uncertain climate sensitivity and damage function estimates and, finally, the uncertain evolution of total factor productivity.

This thesis is laid out as follows. First, Chapter 2 addresses the question whether natural resource revenues should be consumed, saved or invested and examines the factors affecting the magnitude of the precautionary savings incentive in particular. In the same chapter, the framework developed is then applied to three different countries, Norway, Iraq and Ghana, to illustrate the range of different answers to this question. Building on the results in Chapter 2, Chapter 3 performs a case study and shows how these answers may translate to more practical government policy by looking at the Canadian province of Alberta. It is intended to translate the insights from Chapter 2 and the wider literature to policy makers and it does not contain new fundamental economic insights. Chapter 4 extends the framework of Chapter 2 by endogenizing the above-ground portfolio allocation and the below-ground resource extraction decision: resource revenues are no longer exogeneous and the uncertainty of above-ground savings is also taken into account. Using analogous computational methods, Chapter 5 then moves onto new ground by considering the effect of different types of uncertainty on estimates of the social cost of carbon and thereby the optimal price on carbon emissions. Finally, conclusions are drawn and directions for further work given in Chapter 6.

1.3 Symbols and conventions

Although the same symbols are generally used to denote the same quantities throughout this thesis, the definitions of symbols are local to each chapter, unchanged from the corresponding papers and may differ in certain instances. For example, all variables in Chapter 2 are expressed in efficiency units, that is, normalized by economic and population growth.

1.4 Publications

The chapters in this thesis have been published as follows:

- Chapter 2 as Van den Bremer, T.S. & Van der Ploeg, F. (2013) Managing and harnessing volatile oil windfalls. *IMF Economic Review*, **61** (1), pp. 130-167.

- Chapter 3 as Van den Bremer, T.S. & van der Ploeg, F. (2016) Saving Alberta's resource revenues: role of intergenerational and liquidity funds. *Energy Policy*, **99**, pp. 132-146.

- Chapter 4 Van den Bremer, T.S., Van der Ploeg, F. & Wills, S. (2016) The elephant in the ground: managing oil and sovereign wealth. *European Economic Review*, **82**, pp. 113-131.

- Chapter 5 in working paper form as Van den Bremer, T.S. & van der Ploeg, F. (2018) Pricing carbon under economic and climactic risks: leading-order results from asymptotic analysis. *CEPR Discussion Paper* No. DP12642 and as Van den Bremer, T.S. & Van der Ploeg, F. (2018) The risk-adjusted carbon price. *OxCarre Research Paper* No. 203.

Chapter 2

Precautionary saving for resource price uncertainty

This chapter examines the role of precautionary saving in the optimal management of natural resource windfalls. Three funds are necessary to manage such a windfall: intergenerational, liquidity, and investment funds. Precautionary saving for resource price uncertainty drives the size of the liquidity fund. The optimal liquidity fund is larger if the windfall lasts longer, oil price volatility, prudence, and the GDP share of oil rents are high and productivity growth is low. The theoretical insights obtained in this chapter are applied to the windfalls of Norway, Iraq, and Ghana. The optimal size of Ghana's liquidity fund is tiny even with high prudence. Norway's liquidity fund is bigger than Ghana's. Iraq's liquidity fund is colossal relative to its intergenerational fund. Only with capital scarcity, should part of the windfall be used for domestic investment. The chapter illustrates how this can speed up the process of development in Ghana despite domestic absorption constraints.

The contents of this chapter have been published as Van den Bremer, T.S. & Van der Ploeg, F. (2013) Managing and harnessing volatile oil windfalls. *IMF Economic Review*, **61** (1), pp. 130-167.[1]

[JEL E21, E22, D91, Q32]

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2.1 Introduction

Many countries experience substantial oil revenue windfalls[2]. The consensus is that these should be put in a sovereign wealth fund to smooth the benefits across generations, and that these countries should have buffers to cope with oil price volatility. This is sound advice for countries that are well integrated into world capital markets, but not for countries facing capital scarcity or constraints on external borrowing. Oil windfalls should then be used not only to accumulate sovereign wealth but also to invest in the domestic economy and boost development (Collier et al. 2010, van der Ploeg and Venables 2011, 2012). Some countries harness these windfalls for growth and development, especially if institutions are good, but others suffer from poor growth despite large resource bonanzas (e.g. Sachs and Warner (1997), Mehlu et al. (2006), Boschini et al. (2007), van der Ploeg (2011)). Given the political and institutional failures in many oil-rich developing economies, it is a challenge to transform subsoil wealth into productive growth-enhancing physical and human capital. An additional challenge facing oil-rich countries is the high volatility of oil prices given their adverse effects on growth, especially in countries that have poor financial systems, restrictions on international trade and unrestricted capital flows and that are landlocked and ethnically fractionalized (e.g. Blattman et al. (2007), van der Ploeg and Poelhekke (2009), Aghion et al. (2009)).

This chapter's prime objective is thus to address the vexed question how to put oil windfalls to good use and how to cope with the historically high volatility of oil prices.[3] In particular, this chapter sets out to obtain an estimate of the optimal shares of the windfalls to be allocated to intergenerational saving when there is no uncertainty about future oil prices, on the one hand, and precautionary saving, which is what is needed additionally when there is uncertainty about future oil prices, on the other hand, and thus assess how much of the windfall is left to boost consumption. Its second objective is to understand why capital scarcity requires that part of the windfall is spent on domestic investment and consumption (van der Ploeg and Venables 2011, 2012). The empirical observation that fast growth often goes together with reductions in net foreign liabilities - the "allocation puzzle" (e.g. Aizenman et al. (2007), Prasad et al. (2007), Gourinchas and Jeanne (2007)) - might be explained if capital is not invested in high-debt economies because of the higher risk of expropriation.[4] Political economy frictions and the

[2]Whenever this chapter refers to oil, it should be interpreted to refer to natural resources (e.g. gas, diamonds, copper, bauxite, phosphate) more generally and could also be interpreted as remittances (which decline as migrant workers return home or lose connection with the home country) or foreign aid. Oil is thus used as a shorthand for a windfall of foreign exchange and oil price as a shorthand for commodity prices.

[3]Earlier work uses a model of a small open economy that exports exhaustible resources to quantify optimal precautionary saving in response to volatile resource prices and demonstrates that current account balances of countries with a greater weight of resource revenue to future income are bigger (Bems and de Carvalho Filho 2011).

[4]Limited commitment incentivizes the government to pay down external debt along the adjust-

resulting debt dynamics may jointly explain the empirically observed negative relationship between volatility and growth. The model of capital scarcity used in this chapter is inspired by such considerations, but abstracts from micro foundations and simply postulates a relationship between sovereign debt and the risk premium and explores how this affects economic development. The final objective is to allow for the inefficiency of public capital projects, since absorption constraints become binding if the oil windfall is used to rapidly increase public investment.

The private sector cannot achieve the first-best outcome because of various market failures: it does not have good access to international capital markets and may be less able to smooth consumption than the government; derivatives and hedging may be too costly or politically infeasible; it does not internalize the interest-spread externality to do with sovereign risk and capital scarcity; public goods such as infrastructure, education, or health are inadequately provided by the market; the economy may not be able to absorb a rapid build-up of public capital. Furthermore, even if rates of return on domestic capital and foreign assets are equalized, the marginal product of public capital may be higher and the supply of public capital thus lower due to various market and non-market distortions unless the government corrects for these distortions. We use the metaphor of capital scarcity to capture this. There are then two decisions to consider: the first is how much to save; the second is whether to invest at home or abroad.

Three funds are thus necessary to manage the revenues from an oil windfall: *intergenerational* funds to smooth benefits across generations, *liquidity* funds to cope with oil price volatility, *investment* funds to manage domestic investment in case of capital scarcity and absorption constraints.[5] This chapters presents an infinite-horizon, continuous-time framework for optimal management of stochastic oil windfalls and derives how much of the windfall should be allocated to each of these three funds. This chapter shows that the optimal size of the liquidity fund is larger if the windfall lasts longer and oil price volatility, prudence and the GDP share of oil rents are high. With capital scarcity and sluggish adjustment of public capital, part of the windfall must be spent on domestic investment. To illustrate how the importance of each fund depends on the particular features of the economy, the framework developed in this chapter is applied to three very different oil-rich countries: Norway, Iraq, and Ghana. This illustrates why the declining windfall of Norway implies a substantial intergenerational and only a small liquidity fund, why the liquidity fund is the main concern for Iraq, why the optimal liquidity fund for Ghana is tiny and why the real challenge for Ghana is how to use its windfall to invest in the domestic economy and cope with absorption constraints.

ment path (e.g. Cohen and Sachs (1986), Aguiar and Amador (2011)). Debt overhang can exacerbate volatility (Aguiar et al. 2009).

[5] Some share of oil windfall, typically, ends up in private hands, which may not end up in the funds administered by the government or independent authorities. This chapters considers optimal size of the “funds” for the economy as a whole.

This chapter is laid out as follows. First, §2.2 discusses the principles of using sovereign wealth funds to manage and harness oil windfalls and points out some obstacles to using options and structured derivatives to hedge the risk of oil price volatility. This section puts the theoretical framework introduced in this chapter into a practical policy perspective. §2.3 puts forward our theoretical framework to derive the optimal management of debt, assets, and domestic investment in oil-rich economies faced with volatile oil prices and capital scarcity. §2.4 discusses the different types of oil (and gas) windfalls experienced by Norway, Iraq, and Ghana. §2.5 presents the calibration details including those related to oil price volatility, the inefficiency of public investment, and interest premium on foreign debt. §2.6 applies the theoretical framework to estimate the sizes of the intergenerational and liquidity funds as well as the primary deficit increments for the windfalls of Norway, Iraq, and Ghana, respectively. §2.7 discusses the optimal "investing to invest" strategy for Ghana under the assumption of capital scarcity. Finally, §2.8 concludes and offers some policy suggestions.

2.2 Principles of managing volatile oil windfalls

2.2.1 Three types of funds for managing volatile oil windfalls

Oil windfalls last for a limited period, are often known some years in advance, and are highly volatile and unpredictable. Three types of funds can play a role in managing these features of oil windfalls:

1. An *intergenerational* fund to smooth the benefits of a temporary windfall over current and future generations abstracting from uncertainty about future oil prices. During the windfall when oil is produced and sold, the revenue is put into the intergenerational fund. Once the windfall has ceased, the returns on this fund are used to finance the general deficit.

2. A *liquidity* fund to collect additional precautionary buffers as a prudent response to oil price volatility. This fund is designed to self-insure against periods when the oil price is low.

3. An *investment* fund to temporarily park funds until domestic investment projects are ready to be undertaken and to collect any returns from these investments.

There are distinct reasons for having each of these funds (e.g. Collier et al. (2010), van der Ploeg and Venables (2012)). The *intergenerational* fund is used to smooth consumption in the face of foreseen changes in oil revenue. For example, it may be known that current oil reserves will last for another 30 years. The permanent income hypothesis (PIH) then suggests that during the windfall the temporary component of the windfall is saved in foreign assets while the permanent component is used to fund the primary deficit. After 30 years, the interest on accumulated assets finances the same increase in the primary deficit as during the windfall.[6] Hence, through judicious management of foreign assets accumulated in a sovereign wealth fund (that is, the intergenerational fund), the temporary windfall leads to a permanent increase in the primary deficit, which can be split into a permanent increase in consumption or a permanent reduction in taxes depending on political preferences. If the oil windfall is expected to decline, the country must save to achieve the same consumption increment or tax cut in all time periods. If the windfall is anticipated some years ahead, the PIH suggests that the country borrows with the future windfall as security so that the permanent increase in the primary deficit extends to the announcement period. The so-called bird-in-hand (BIH) rule, which states that all revenues are put in a fund, and only a certain percentage of this fund is used for consumption every year, rules out using oil windfalls as collateral and is thus a prudent variant of the permanent income rule. For example, Norway puts all oil revenue in its intergenerational fund and takes out 4 percent each year to finance the general deficit.

A *liquidity* fund is used to cope with oil price volatility. This fund is larger if oil prices are more volatile (higher standard deviation of oil price shocks as fraction of planned consumption), oil price shocks are more permanent, policy makers more prudent, and the windfall lasts for a longer period. Prudence requires a positive third derivative of the utility function. There may also be precautionary saving in response to asset return uncertainty, but only if there is a motive for intergenerational saving or borrowing in the first place (see Chapter 4).

Finally, an *investment* fund is necessary if countries are not well integrated in global capital markets. In countries with perfect access to world capital markets no part of the windfall should be spent on domestic investment projects: the windfall should feed the intergenerational and liquidity fund and curb the general deficit, but not feed an investment fund. However, many developing countries suffer from capital scarcity and pay a premium when borrowing to fund investment projects. It is then optimal to spend part of the oil windfall on domestic investment to alleviate capital scarcity (van der Ploeg and Venables 2011). Domestic investment projects may face all kinds of absorption, planning and legal constraints in which case it makes sense to temporarily park part of the oil windfall until it is feasible to undertake the project.

[6] This is related to the celebrated Hartwick rule, which says that oil rents should be saved, so that exhaustible assets under the ground are fully transformed into assets above the ground (Hartwick 1977).

2.2.2 What assets should the intergenerational and liquidity funds invest in?

By careful choice of the sovereign wealth portfolio an oil-rich country can hedge oil income risk. The key question is whether one should choose equity holdings in companies whose fortunes move inversely with the world price of oil or in companies who are not affected by or benefit from increases in the oil price. Examples of the former are intensive energy users such as aluminium smelters, steel producers, oil companies, while examples of the latter are companies that offer substitutes for fossil fuels, such as the production of energy-efficient cars. Net asset holders that invest in companies whose share prices vary inversely with the price of oil need to hold smaller precautionary buffers. This question is addressed in detail in Chapter 4.

2.2.3 Hedging against volatile oil prices

An alternative way for an oil exporter to deal with the volatility of future oil prices is to hedge and transfer the risk to those who are better able to bear it (e.g. Stulz (2002)). For example, Mexico bought a put option at a strike price of $140 per barrel in 2009 after oil prices reached heights of almost $140 per barrel in 2008.[7] When the oil price went significantly below this strike price, Mexico exercised the option and collected $8 billion. The costs of the option were $1.5 billion. The drop in oil revenue was compensated, to a large extent, by the profits on the option. Options could thus offer an insurance policy against the risk of future oil price volatility. Ecuador, Columbia, Algeria, Texas, and Louisiana have also used options to protect themselves against volatile oil and gas prices.

Such plain-vanilla put options are costly. Lu and Neftci (2008) therefore argue for structured-reverse options that lower the cost of plain-vanilla options by selling other options simultaneously (zero-premium collar). Yet, such products can lead to substantial losses if commodity prices rise above their cap. Barrier options (for example an up-and-out put option or a knockout option) are cheaper. Various developing countries use commodity derivatives markets to hedge against commodity price risk (e.g. Larson et al. (1998)). Options and other structured derivative products can help manage oil price volatility, but are expensive and risky themselves. For most commodities (including oil) maturities are too short and financial markets too thin to provide adequate protection. There are also large political risks if a lot of money has been spent and options are not exercised. If the option is exercised with profit, this may be denounced as speculation. Big commodity exporters that hedge can influence the market price, especially if they have private information, and also stand to be accused of speculation rather than insurance. Liquidity funds could offer a more attractive alternative.

[7] All dollars ($) in this chapter refer to U.S. dollars.

2.2.4 General economic policy

More generally, flexible goods, labour and capital markets help to deal with oil price volatility. It also helps to avoid irreversible commitments, which cannot be kept if oil prices fall by a large amount. Independent liquidity funds reduce the need for such politically difficult measures. Further, it helps to diversify into sectors whose fortunes are orthogonal to or negatively correlated with the commodity sector. The government could also relate debt payments to the oil price to protect itself against oil price volatility. The idea is that, in the event of a crash in oil prices, the governments debt burden would fall too. Governments may also help the private sector to hedge against oil price volatility and prevent changes in world oil prices being passed on fully to domestic consumers, especially if households are risk averse and face high adjustment costs, credit markets and self-insurance are imperfect and hedging opportunities for private individuals using futures contracts and options are limited (e.g. Federico et al. (2001)).[8] The trade-off between retail oil price stability and government fiscal stability poses important challenges, but is beyond the scope of the present thesis.

2.3 Theory of managing volatile oil windfalls

To make a quantitative assessment of the size of the intergenerational and liquidity funds and the optimal amount of domestic investment to undertake in response to a temporary and volatile oil windfall, this chapter formulates a simple welfare-based, infinite-horizon, continuous-time model of an oil-exporting economy.[9] This chapter also allows for capital scarcity and capital adjustment costs for investment. Finally, this chapter incorporate the effects of growth by including exogenous productivity and population growth trends. The trend rate of growth equals the sum of the growth rates of population and labour augmenting technical progress, denoted by n and g, respectively. All variables in this chapter are expressed in efficiency units (that is, divided by $\exp((n+g)t)$). The instantaneous utility of per-capita consumption at time t is given by the utility function:

$$U(C(t)e^{gt}) = \begin{cases} \dfrac{(C(t)e^{gt})^{1-\eta} - 1}{1-\eta} & \text{for } \eta \neq 1, \\ \log(C(t)) + gt & \text{for } \eta = 1, \end{cases} \tag{2.3.1}$$

where $1/\eta$ denotes the elasticity of intertemporal substitution and C aggregate consumption in efficiency units. The coefficient of relative risk aversion equals

[8]In many poor countries the share of petroleum consumption in household income is high, income and price elasticities for petroleum demand are low and households are relatively risk averse, in which case the risk aversion effect dominates the effect of substituting away from petrol if its price is high and towards petrol if its price is low, so that consumers benefit from petrol price stability (e.g. Turnovsky et al. (1980)).

[9]Bems and de Carvalho Filho (2011) offer a discrete-time approach for dealing with oil price uncertainty and precautionary buffers, but do not deal with capital scarcity and productivity growth.

$CRRA \equiv -CU''(C)/U'(C) = \eta$ and also corresponds to the coefficient of relative intergenerational inequality aversion in this formulation. It is thus crucial in determining the trade-offs between present and future consumption. The coefficient of relative prudence is $CRP \equiv -CU'''(C)/U''(C) = 1 + \eta$. Public investment is denoted by I and is subject to internal adjustment costs, so that the price of public investment goods equals $1 + (1/2)\phi I/S$ with $\phi > 0$ the adjustment cost parameter and S the stock of public capital. One interpretation is that, when investment is increased rapidly, the price of investment goes up as a result of various absorption constraints. Denoting the depreciation rate of the public capital stock by $\delta^* > 0$, the accumulation of the public capital stock in efficiency units is written as:

$$\frac{dS(t)}{dt} = I(t) - \delta S(t), \quad S(t = 0) = S_0, \tag{2.3.2}$$

where $\delta \equiv \delta^* + n + g$ is the effective depreciation rate (including the trend rate of growth). Uncertainty in trend growth is not considered. Production by the representative firm (indicated by superscript f) operating under constant returns to scale is given by:

$$Y^f(t) = F\left(K^f(t), S(t)/Y(t)\right) = (A_0^*)^{1-\alpha}(K^f(t))^{\alpha}[S(t)/Y(t)]^{\beta^*}, \tag{2.3.3}$$

where A_0^* is the level of labour-augmenting technical progress, $0 < \alpha < 1$ the share of private capital in aggregate production and $\beta^* > 1$ the marginal effect of public capital as a share of aggregate production on firm-level output. Public capital thus boosts private production. Private capital is financed from abroad at a user cost of capital equal to the exogenous world interest rate r^* plus the depreciation rate of private capital δ^P, hence $\alpha Y^f/K^f = r^* + \delta^P$. The intensive-form production function with this private sector response substituted into (2.3.3) is in symmetric equilibrium with $Y = Y^f$ and $K = K^f$ and is equal to:

$$Y(t) = A_0 S(t)^{\beta}, \quad A_0 \equiv \left[A_0^*\left(\frac{\alpha}{r^* + \delta^P}\right)^{\alpha}\right]^{\frac{1}{1+\beta^*-\alpha}}, \ \beta \equiv \frac{\beta^*}{1+\beta^*-\alpha} > 1. \tag{2.3.4}$$

Equation (2.3.4) states that total factor productivity and public investment act to increase private output.

The asset accumulation equation for the oil-exporting economy gives the increase in sovereign wealth as growth-corrected interest income $(r + \Pi)B$ plus oil revenue $(P - \Psi)O$ plus (non-oil) production income Y minus consumption and the cost of public investment (including the costs of absorption):

$$\frac{dB(t)}{dt} = \left(r + \Pi(B)\right)B(t) + \left(P(t) - \Psi\right)O(t) + Y(t) - C(t) - I(t) - \frac{1}{2}\phi\frac{I(t)^2}{S(t)}, \tag{2.3.5}$$

where $r \equiv r^* - n - g$ denotes the growth-corrected world interest rate, Π the risk premium on borrowing from abroad, B the stock of foreign assets held by the country, P the exogenous world price of oil, Ψ the constant extraction cost per barrel of oil, and O the volume of oil production. Equation (2.3.5) thus gives the current account dynamics of the economy and is solved subject to the initial condition $B(0) = B_0$. Since variables are measured in efficiency units, the world interest rate r^* is corrected for the trend rate of economic growth. Without capital scarcity and with accumulated sovereign wealth large enough, the risk premium Π is zero. If the economy is a substantial borrower from abroad (i.e. $B < 0$), it has to pay an interest premium, and this premium rises with indebtedness (in efficiency units), i.e. $\Pi > 0$ and $\Pi'(B) < 0$. For a given level of real indebtedness for the economy as a whole, the premium the country has to pay is thus less if there are more people to share the burden of the debt and if the ability to pay as proxied by the state of technical progress is higher. One could relate the size of the risk premium to the economy's ability to pay by specifying $\Pi(B/Y)$ or to the size of the anticipated windfall. Since there is no conclusive empirical support for oil windfalls alleviating the debt premium paid on international capital markets, such an alleviating effect is not taken into account.

Policy makers face various types of uncertainty: about oil prices, reserves, investment returns, asset returns, and general economic outcomes, notably growth prospects. This chapter focusses on the most important form of uncertainty for oil-rich economies, i.e. oil price volatility. The oil price is first described by a geometric Brownian motion:

$$dP(t) = \upsilon_P P(t)dt + \sigma_P P(t)dW(t) \quad \text{or} \quad d\log(P(t)) = \left(\upsilon_P - \frac{1}{2}\sigma_P^2\right)dt + \sigma_P dW(t), \tag{2.3.6}$$

where $W(t)$ is a Wiener process satisfying $W(t) - W(s) \sim N(0, t-s)$ for $t \geq s$. The constants υ_P and σ_P are the percentage drift and the percentage volatility, respectively. The solution to (2.3.6) is:

$$P(t) = P_0 \exp\left(\left(\upsilon_P - 0.5\sigma_P^2\right)t + \sigma_P W(t)\right), \tag{2.3.7}$$

with expectation and variance $\mathrm{E}[P(t)] = P_0 e^{\upsilon_P t}$ and $\mathrm{var}[P(t)] = P_0^2 e^{2\upsilon_P t}\left(e^{\sigma_P^2 t} - 1\right)$, respectively. To reflect the strong effect of even a small degree of mean reversion on the propagation of shocks, the mean-reversion model of Schwartz (1997) for the oil price is also considered:[10]

$$dP(t) = \left(\eta_P(m_P + \upsilon_P t - \log(P(t))) + \upsilon_P\right)P(t)dt + \sigma_P P(t)dW(t), \tag{2.3.8}$$

which can be rewritten as a homoskedastic AR(1) process for $\log(P(t))$:

$$d\log(P(t)) = \left(\eta_P(m_P^* + \upsilon_P t - \log(P(t))) + \upsilon_P\right)dt + \sigma_P dW(t), \tag{2.3.9}$$

where $m_P^* = m_P - \sigma_P^2/2\eta_P$ (from Itō calculus).

[10]Thus avoiding the heteroskedasticity problems, which would arise when estimating a geometric Ornstein-Uhlenbeck process (Dixit and Pindyck 1994), and exploiting the fact that (2.3.8) can be written as a standard homoskedastic AR(1) process after a logarithmic transformation.

The government thus maximizes the expected value of a utilitarian social welfare function, i.e. the sum of the utilities of per-capita consumption (2.3.1) over all members of the population:

$$\mathrm{E}_0\left[\int_0^\infty U\left(C(t)e^{gt}\right)e^{(n-\rho^*)t}dt\right] = \begin{cases} \mathrm{E}_0\left[\int_0^\infty \frac{C(t)^{1-\eta}-1}{1-\eta}e^{-\rho t}dt\right] \text{ for } \eta \neq 1, \\ \mathrm{E}_0\left[\int_0^\infty (\log(C(t)) + gt)e^{-\rho t}dt\right] \text{ for } \eta = 1, \end{cases} \tag{2.3.10}$$

subject to the public capital accumulation equation (2.3.2), the intensive-form production function (2.3.4), the asset accumulation equation (2.3.5) and the stochastic dynamics of the oil price (2.3.6) or (2.3.8). The social rate of discount is denoted by $\rho^* > 0$. Since social welfare (2.3.10) is in terms of consumption in efficiency units, the social discount rate is corrected for population growth and (depending on the income and substitution effect) for labour-augmenting technical progress: $\rho \equiv \rho^* - n - (1-\eta)g$.

Using Itō calculus to solve this stochastic optimization problem, the following conditions can be obtained:[11]

$$\frac{1}{dt}\mathrm{E}_t[dC] = \frac{1}{\eta}C\Big(r + \Pi(B) + \Pi'(B)B - \rho\Big) + \frac{1}{2}CRP\left(\frac{\partial C}{\partial P}\right)^2\left(\frac{\sigma_P P}{C}\right)^2 C, \tag{2.3.11}$$

$$\frac{dS}{dt} = \left(\frac{1}{\phi}(q-1) - \delta\right)S, \tag{2.3.12}$$

$$\frac{1}{dt}\mathrm{E}_t[dq] = (r + \Pi(B) + \Pi'(B)B + \delta)\,q - (1-\alpha)\beta A_0 S^{\beta-1} - \frac{1}{2\phi}(q-1)^2, \tag{2.3.13}$$

$$\frac{dB}{dt} = A_0 S^\beta + (P - \Psi)O + (r + \Pi(B))\,B - C - \frac{1}{2\phi}(q^2-1)S, \tag{2.3.14}$$

where q is the social cost of capital, $\partial C/\partial P$ is the effect of an oil price shock on consumption (cf. 'marginal propensity to consume' out of the wealth generated by an oil price shock) and $CRP = 1 + \eta > 1$ is the coefficient of relative prudence. Equations (2.3.11-2.3.14) must be solved subject to the initial conditions $C(0) = C_0$, $S(0) = S_0$, $q(0) = q_0$ and $B(0) = B_0$, where C_0 and q_0 are 'free' or forward-looking variables. Oil production in efficiency units $O(t)$ is exogenous.

Equation (2.3.11) is a modified version of the Keynes-Ramsey rule, which states that the growth rate in consumption is proportional to the social cost of borrowing minus the social discount rate. More intergenerational inequality aversion (higher η) implies a smaller growth rate to avoid inequality between present and future generations. The social cost of borrowing corresponds to the world interest rate plus interest premium plus the term $\Pi'(B)B$ to correct for and internalize the

[11] These optimality conditions extend the results of van der Ploeg (2012) to a stochastic setting.

interest spread externality. For an economy with capital scarcity, it is thus optimal to have a rising path of consumption: the economy consumes less upfront to pay off debt and lower the risk premium. Equation (2.3.11) also contains a prudence term (see §2.3.1 for discussion). Equation (2.3.12) describes the public-sector capital stock dynamics, where the rate of public investment is proportional to its social value, $I/S = (q-1)/\phi$. Equation (2.3.13) gives the intertemporal efficiency condition for public-sector investment: the marginal product of public capital plus the marginal reduction in adjustment cost must equal the social cost of borrowing (the market interest rate plus the interest premium on government debt Π plus the correction term to allow for the rising cost of public debt $\Pi'(B)B$ plus the depreciation rate δ). Finally, (2.3.14) gives the dynamics of sovereign wealth with the cost of public-sector investment and output substituted in. It supposes that the country has linear oil extraction costs and does not have monopoly power on the world market.

The five-dimensional system (2.3.6) or (2.3.8) and (2.3.11-2.3.14) has three predetermined state variables, P, B and S, and two non-predetermined variables, C and q. Hence, C_0 and q_0 adjust instantaneously to ensure that the economy is on its stable manifold and thus satisfy a corresponding transversality condition on public debt and capital. Two cases are now considered in turn: the case without capital scarcity ($\Pi = 0$) and the case without oil price volatility ($\sigma_P = 0$).

2.3.1 No capital scarcity: intergenerational and liquidity funds

Consider an economy with good access to world capital markets and no interest premium on national borrowing, $\Pi = 0$. The so-called separation theorem for public investment then holds: it is suboptimal to spend part of the oil windfall on public investment. How much public investment should be undertaken does not depend on the oil revenue coming in, but only on the costs and benefits of public investment itself. Any financing need is supplied by international capital markets. To see this, note that with $\Pi = 0$, (2.3.12) and (2.3.13) are decoupled from the rest of the economy (i.e. independent of (2.3.6) or (2.3.8), (2.3.11) and (2.3.14)) and thus independent of the size of the oil windfall. This is a lesson often forgotten in highly developed oil- or gas-rich economies such as Norway and the Netherlands that continues to hold even with the recent sovereign debt crisis. We assume that the dynamics of (2.3.12) and (2.3.13) have played out, so that the social value of public capital, the stock of public capital and aggregate production are independent of the size of the windfall:

$$q = 1 + \phi\delta,\ S = \left[\frac{(1-\alpha)\beta A_0}{r+\delta+\phi\delta(r+\delta/2)}\right]^{\frac{1}{1-\beta}},\ Y = A_0^{\frac{1}{1-\beta}}\left[\frac{(1-\alpha)\beta}{r+\delta+\phi\delta(r+\delta/2)}\right]^{\frac{\beta}{1-\beta}}. \tag{2.3.15a,b,c}$$

For purposes of optimally managing an oil windfall these variables can thus be treated as exogenous. Equations (2.3.12), (2.3.13) and (2.3.14) then lead to the following asset accumulation equation:

$$\frac{dB}{dt} = Y - \left(\delta + \frac{\phi\delta^2}{2}\right)S + (P - \Psi)O + rB - C, \tag{2.3.16}$$

subject to $B(0) = B_0$. The first two terms indicate output net of costs (including absorption costs) of public investment. Furthermore, oil price volatility induces an additional precautionary savings response as can be seen from (2.3.11), which reduces to:

$$\frac{1}{dt}\mathrm{E}_t\left[dC\right] = \underbrace{\frac{1}{\eta}\left[r - \rho\right]C}_{\text{growth effect}} + \underbrace{\frac{1}{2}CRP\left(\frac{\partial C}{\partial P}\right)^2\left(\frac{\sigma_P P}{C}\right)^2 C}_{\text{prudence effect}}, \tag{2.3.17}$$

As $r = \rho - g\eta$, if $r^* = \rho^*$, the first term on the right-hand side of (2.3.17) sums up the *growth* effect. In a growing economy, it is optimal to borrow if prospects are good. The second term in (2.3.17) is the *prudence* effect. With volatile oil prices the expected time path of consumption slopes upwards. Precautionary saving implies consumption is initially low, especially if the coefficient of relative prudence and oil price uncertainty as share of consumption (i.e. $\sigma_P P/C$) are high. Under the permanent income hypothesis, countries with a temporary windfall save more of their windfall than those with a more permanent windfall, and thus have a smaller prudence effect as can be seen from the leading-order partial derivative term[12]:

$$\frac{\partial C(t)}{\partial P(t)} = \left(r - (r - \rho)/\eta\right)\int_t^\infty \frac{\partial \mathrm{E}_t\left[P(\tau)\right]}{\partial P(t)} O(\tau)e^{-r(\tau-t)}d\tau, \tag{2.3.18}$$

where $r - (r - \rho)/\eta$ is the marginal propensity to consume out of wealth and from (2.3.6) or (2.3.8):

$$\frac{\partial \mathrm{E}_t\left[P(\tau)\right]}{\partial P(t)} = \begin{cases} 1 & \text{random walk,} \\ \frac{\mathrm{E}_t[P(\tau)]}{P(t)}e^{-\eta_P(\tau-t)} & \text{AR(1),} \end{cases} \tag{2.3.19}$$

for $\tau \geq t$. Equation (2.3.19) can be made more explicit:

$$\frac{\partial \mathrm{E}_t\left[P(\tau)\right]}{\partial P(t)} = \begin{cases} 1 & \text{random walk,} \\ \dfrac{e^{m_P\left[1-e^{-\eta_P(\tau-t)}\right]+\log(P(t))e^{-\eta_P(\tau-t)}}}{P(t)}e^{-\eta_P(\tau-t)} < 1 & \text{AR(1).} \end{cases} \tag{2.3.20}$$

If $\eta_P = 0$, shocks are permanent and all expected future prices increase by the same amount as the initial shock. If $\eta_P = \infty$, shocks are purely transitory and have zero effect on future expected oil prices. In general, mean reversion implies that the effect of current price shocks on expected values of future price shocks is positive

[12]Foregoing a formal introduction of the perturbation scheme, we assume oil price volatility is small, so that a leading-order approximation to the partial derivative $\partial C/\partial P$ can be obtained from the deterministic solution.

but less than one. Hence, the marginal propensity to consume future consumption out of a current shock to the oil price (2.3.18) is less with mean reversion, and thus, from the Euler equation (2.3.11) or (2.3.17), the effect on precautionary saving is smaller too. A higher trend growth rate cuts oil production in efficiency units and thus, from (2.3.18), depresses the marginal propensity to consume and from (2.3.11) or (2.3.17) also reduces precautionary saving. In effect, there is less need for precautionary saving if productivity growth makes you richer in the future and hence better able to deal with future income shocks.

Using the leading-order approximation for the partial derivative term in (2.3.20) (cf. the more formal introduction of a perturbation scheme in the volatility in Chapter 1), we solve the resulting initial value problem consisting of two simultaneous ordinary differential equations (2.3.16-2.3.17) with a standard multiple-shooting algorithm.

2.3.2 Capital scarcity: investing to invest

To capture the fact that developing economies experience capital scarcity and may have substantial sovereign debt before enjoying an oil bonanza, we suppose that countries pay a risk premium on their sovereign debt and take this as a metaphor for capital scarcity (cf. van der Ploeg and Venables (2011, 2012)). The separation theorem no longer holds as the optimal level of domestic investment now depends on the level of saving and the size of the windfall of foreign exchange. The relevant social cost of borrowing exceeds the private cost of borrowing, since the government, in contrast to the private sector, internalizes the higher cost of borrowing resulting from having debt. This results in a corresponding increase in the cost of public investment and explains why the separation theorem breaks down.

With capital scarcity both present and future consumption are lower even without windfall uncertainty. Further, present consumption is lower than future as capital scarcity prevents public debt from being raised to sufficiently high levels to fully smooth consumption. The higher social cost of borrowing holds back public investment and thus economic development. With volatile windfall income present and future consumption diverge even further, because public debt is cut and public investment raised for precautionary reasons. The precautionary buffer strikes the optimal balance between prudence and intergenerational equity.

The effects of investment returns uncertainty, not considered in the model, obviously also matters. Without capital scarcity, uncertain returns on public investment make it prudent to redeem more debt, which depresses present consumption and increases future consumption. For high degrees of investment returns uncertainty future consumption becomes smaller again due to decreasing returns to public investment. Capital scarcity lowers public investment, debt and average consumption and forces a wedge between present and future consumption even if there is no investment returns uncertainty. With investment returns uncertainty, prudence acts to shift more income to the future. The drop in public investment is stronger if there is no capital scarcity, because the initial investment level is higher. With capital scarcity, saving is more powerful. It curbs both the debt and the interest to be paid on it, so the country needs to save less. Uncertainty about public investment returns thus cause countries to save more and invest less. The view that oil-rich countries facing capital scarcity tend to be big savers and small investors (Cherif and Hasanov 2011) also accords well with an economy without capital scarcity and high degrees of uncertainty about returns on investment.

2.4 Three different windfalls

2.4.1 Norway: declining oil windfall, no capital scarcity

Norway discovered its first oil field Ekofisk (one of the world's largest offshore oil basins) in 1969 and started production in 1971. Production has shifted from oil and other liquids to gas. The oil and gas industry constitutes about a quarter of GDP and half of exports. The peak of oil production was at around the turn of the millennium. Government revenue comes from various sources: ordinary and special tax rates on value added, which have been quite volatile with special taxes taking over from ordinary taxes in importance since the early 1990s (together almost 35 percent of value added); net cash flow from the state's direct financial interest in the oil and gas industry (after initial investment outlays of up to 20 percent of value added in the mid 1980s, net return is now more than one fifth of value added); production fees (no longer an important source); dividends from Statoil (about 3 percent of value added).

The Norwegian government puts this revenue in a fund called the Government Pension Fund Global, which started receiving funds in 1996 and has since grown rapidly in size.[13][14] Its aim is to counter the fall of expected petroleum income and

[13]The fund comprises two separately managed funds. The main fund is the Government Pension Fund Global renamed 1 January 2006 and is part of the Norwegian Central Bank (formerly The Government Petroleum Fund established in 1990 and receiving money since 1996). It manages the surplus wealth produced by Norwegian petroleum income (taxes and licenses) and is the second largest pension fund in the world. Since 1998 the fund was allowed to invest up to 40 percent of its assets in the international stock market (60 percent from 2007). The other fund is the Government Pension Fund Norway renamed 1 January 2006 (formerly The National Insurance Scheme Fund established in 1967) and is much smaller.

[14]The 1983 Tempo Committee recommended to convert assets under the ground into a fund and to decouple oil and gas income from spending. The 1988 Steigum Committee advised that public income should depend on the permanent income of total oil and gas wealth (i.e. the value of in-situ oil and gas plus the fund). Norway wanted a pragmatic, operational and easy-to-understand policy rule, which requires credible, robust estimates of future, unproduced oil and gas revenues and the need to avoid political manipulation of forecasts of future oil prices. Since smoothing of consumption, public spending and taxes ex ante may require large variations in the net liabilities or asset position in response to changes in the relevant present values, actual policy was more driven by bird-in-hand

smooth the disrupting effects of oil price volatility, so it has elements of both an intergenerational and a liquidity fund. Roughly 4 percent per year from the fund is used to finance public spending or tax cuts. This 4-percent rule was implemented in 2001 and allows Norway to spread oil and gas revenues to future generations.[15][16] Since future oil revenue cannot be used as collateral for borrowing, Norway's fiscal rule resembles a bird-in-hand rule. Since Norway's budgetary policies take account of declining oil revenue, it also has features of a permanent income rule. Given that Norway's economy is well integrated into world capital markets, capital scarcity is not an issue.

Production from proven oil and gas reserves is expected to fall substantially during the next twenty-five years. Even allowing for improved recovery, discoveries of new fields and undiscovered resources, forecasts show a decline in oil production levels. Projected net oil and gas cash flows to the government decline up to 2060 and are sensitive to the projected oil price. We take the annual investment cost of the entire Norwegian oil and gas sector including exploration costs as a measure of extraction costs. Using historical data for 1970-2010 from Norwegian Petroleum Directorate (2011) we find that, apart from the initial years 1970-75 when extraction costs were still very high as the very first exploratory and extraction activity took place, average extraction costs were \$9 per b.o.e. (barrel of oil equivalent) in the period 1990-2000, \$6 per b.o.e. for 2000-2005 and \$14 per b.o.e. for 2005-2010. For the future, we adopt constant extraction costs of \$15 per b.o.e.

The Norwegian Petroleum Directorate (2011) provides a range of total reserve estimates between 31.5 and 66.7 billion b.o.e. with an average of 46.6 billion b.o.e. at the end of 2011. Because of their different prices and price behaviour, we distinguish between oil[17] and gas reserves: 24.5 billion b.o.e. of gas and 22.0 billion b.o.e. of liquids (mainly oil and henceforth denominated as oil). For the extraction

than permanent-income considerations. This might also have been an appropriate response to deal with oil price volatility.

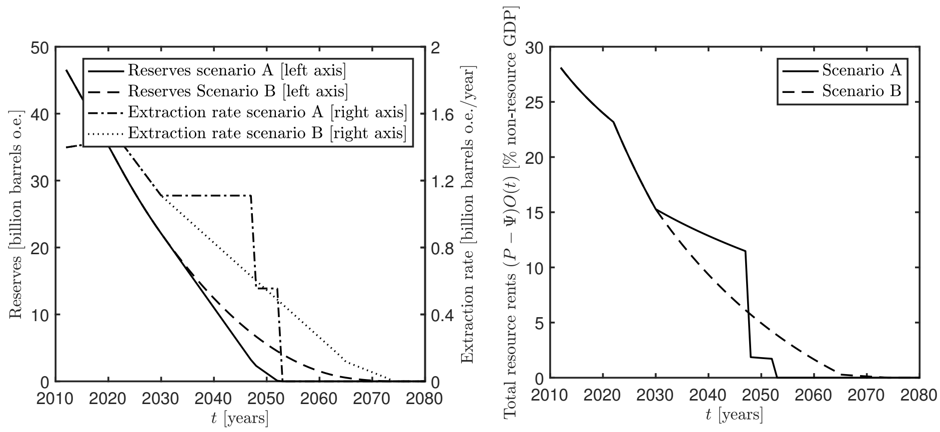
[15]The Fund also allows Norway to stabilize the economy across the business cycle, since the 4%/year is meant to be an average over the business cycle. Since value of the Fund varies with world asset markets, the government has the discretion to deviate from the 4-percent-rule when it deems this necessary.

[16]Norway's fiscal rule for the non-resource primary public-sector deficit has been estimated for 1954-2007 using official forecasts for the expectations on future oil/gas revenue to calculate permanent values by Harding and van der Ploeg (2013). The permanent income hypothesis implies values of 0 and 1 for the coefficients on current and permanent oil/gas revenue and is rejected by the data: the estimated rule suggests that a third (not zero) of each dollar of oil and gas revenue is used to raise the primary public-sector deficit and for permanent oil and gas revenue the effect is 0.3, not 1 (Harding and van der Ploeg 2013). The bird-in-hand hypothesis implies a zero coefficient on permanent oil and gas income, as only already accumulated assets should affect spending decisions. Hence, the estimated rule has features of both the permanent-income and bird-in-hand rule.

[17]We group oil and other liquids only excluding gas as liquids, henceforth referred to as oil, which follows the definition of the Norwegian Petroleum Directorate (2011).

Figure 2.1: Reserves, extraction rates and oil and gas rents for Norway.

(a) Reserves and extraction rates (b) Resource rents



Projected combined oil and gas reserves levels (panel a, left axis), extraction rates (panel a, right axis) (both not in efficiency units) and corresponding resource rents as a share of non-resource GDP (panel b) for Norway based on official forecasts until 2030 and two scenarios for after 2030: (A) constant extraction rate (base case) and (B) linear decline. See §2.4.1 for details.

scenario we use official forecasts by the Norwegian Petroleum Directorate (2011), which are available until 2030 and are shown in the left panel of Figure 2.1.

Thereafter, we consider two scenarios for the extraction rate: (A) continue at 2030 rate until resources are fully exhausted so that gas reserves are exhausted in 2052 and oil reserves in 2048 (base case); (B) linear decline from 2030 until resources are fully exhausted, so that gas reserves are exhausted in 2075 and oil reserves in 2064. The corresponding resource rents are plotted in the right panel of Figure 2.1. Annual resource rents are thus estimated to be \$92 billion/year in 2012 or 28% of non-resource GDP and taper off to 15% of non-resource GDP in 2030 under the official production forecast and our estimated process for the oil price (see §2.5.2).

The size of the Norwegian Government Pension Fund Global (A_0) was a staggering 3,312 billion NOK at the end of 2011 (Norwegian Petroleum Directorate 2011) corresponding to \$592 billion or 122% of GDP and 181% of non-resource GDP. Norwegian GDP in 2011 was 2,720 billion NOK/year or \$484 billion/year and non-resource GDP was \$326 billion/year.

2.4.2 Iraq: huge and long-lasting oil windfalls

The reduction in conflict and ensuing economic progress in Iraq has led to a large increase in oil extraction over recent years.[18] Oil reserves at the end of 2011 were 143 billion barrels of oil, which has been strongly revised upwards from the estimate of 115 billion barrels of oil at the end of 2010. Production is projected to reach 2 billion barrels per year in 2030, up from 1 billion barrels in 2011 (BP 2012). We follow this forecast and let extraction grow linearly until 2030. Thereafter, we consider two scenarios: (A) continued extraction at the 2030 rate until exhaustion, which will take place in 2088 (base scenario); (B) continued extraction at the 2030 rate until 2050 followed by linear decline until exhaustion, which will take place in 2126. Based on the estimated extraction costs for Norway of $4-14 per b.o.e. (part of which is offshore), we guess extraction costs for Iraq, onshore and generally cheap to extract, to be $10 per barrel. The resulting reserves, extraction rates and oil rents are plotted in Figure 2.2. Resource rents start at a staggering 650% of non-resource GDP in 2012 and then taper off in both scenario A and B.

In 2010 oil production accounts for over half of GDP and 83% of government income (IMF 2010) and these rates are even higher at current high oil prices. With estimated extraction costs of $10 per barrel and oil prices at $110 per barrel in 2011, oil rents reach a staggering 87% of total GDP ($115.4 billion/year, 2011 GDP) or 650% of non-resource GDP. Figure 2.2 indicates that oil rents are projected, at an oil price of $110 per barrel, to rise to $200 billion/year in 2030.

2.4.3 Ghana: temporary, small oil windfall

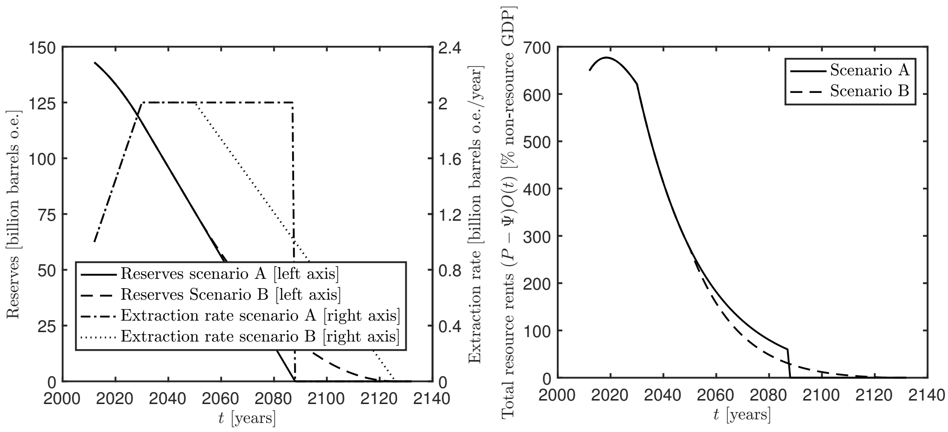
Despite a long history of exploration, oil production in Ghana has been negligible until it started producing oil from the offshore Jubilee field in 2010. Based on current proven reserves, production from the Jubilee field is expected to peak from 2013-2015 at 120,000 barrels of oil per day, and last for 20 years (Tullow Oil 2011). This has the potential to generate up to $1.8 billion per year at peak production. Ghana's potential reserves are 4 billion barrels.[19] Production from the Jubilee

[18]From 1980 to 1988 Iraq was embroiled in the war with Iran. In 1990-1991 the invasion of Kuwait led to the first Gulf War. After a decade of relative peace, 2003 saw the beginning of the second Gulf War with many casualties and about five million refugees. In late 2011 the United States withdrew the last of its troops, but sectarian violence and homicides in Baghdad remain widespread. Nevertheless, Iraq has made significant economic progress since the Transitional Government was established in 2005. Annual headline inflation has fallen from over 60% to single digits, and the dinar has remained stable against the dollar. Debt sustainability has improved with the 80% reduction of the Paris Club in 2004, and negotiations are under way with non-Paris Club creditors. Domestic fuel subsidies have been ended.

[19]Ghana is approximately 50$^{\text{th}}$ in the world in terms of proven oil reserves, much lower than those of Saudi Arabia (265 billion barrels), Canada (175 billion barrels), Venezuela (98 billion barrels) and Nigeria (38 billion barrels). At 160 barrels of oil per person, Ghana's deposits are far less than those of Kuwait (40,000), Saudi Arabia (10,000), Venezuela (3,500) and Nigeria (240). Ghana's reserves per dollar of GDP are approximately 15$^{\text{th}}$ in the world, on par with Angola and Nigeria.

Figure 2.2: Reserves, extraction rates and oil rents for Iraq.

(a) Reserves and extraction rates (b) Resource rents



Projected oil reserves levels (panel a, left axis), extraction rates (panel a, right axis) (both not in efficiency units) and corresponding resource rents as a share of non-resource GDP (panel b) for Iraq. See §2.4.2 for details.

field at its peak will generate up to 30% of the government's income, if oil is at \$75 per barrel, but only for a short period. Ghana has a GDP per capita of \$1,470/year.

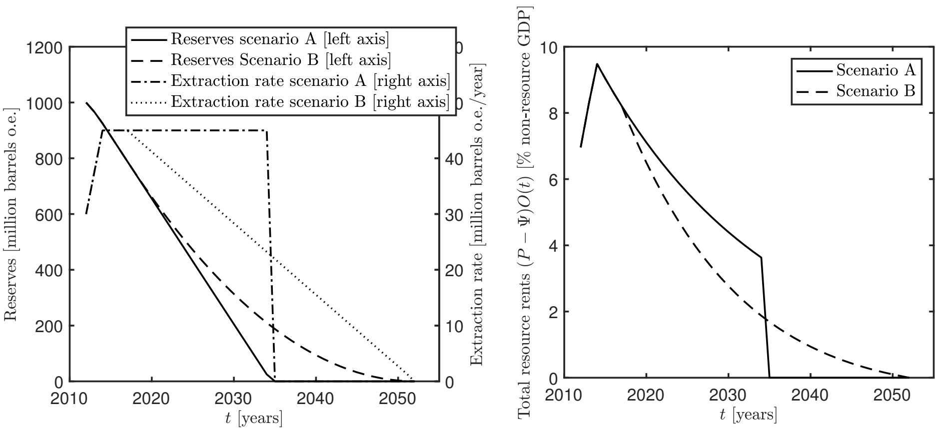
Ghana adopted the Petroleum Revenue Management Act (PRMA) in 2011[20]. It specifies how oil revenues are to be allocated between the annual budget and sovereign wealth funds as the income is received. The government's oil revenue from the Jubilee field has four components: (1) a royalty of 5 percent of gross oil revenues; (2) 13.75 percent of the field's commercial net profits go to the Ghana National Petroleum Corporation (GNPC); (3) an "additional oil entitlement" of 1025 percent of petroleum revenue (net of royalties and the GNPC interest), which accrues if the project rate of return is between 18 and 33 percent; (4) a company income tax of 35 percent on all net profits.

In contrast to Norway, Ghana has only recently enjoyed an oil windfall. The base case extraction scenario used in a comprehensive report by the World Bank is based on proven recoverable oil reserves of 490 million barrels (Dessus et al. 2009) and probably underestimates Ghana's future production and projects an oil

[20]The PRMA allocates government oil revenues between the annual budget and sovereign wealth funds. The allocation is based on "benchmark revenue", using a seven-year moving average of oil prices (including three projected years), and a three-year average of output (including projections for the next year). From the benchmark revenue 50-70% is allocated to the annual budget, of which a minimum of 70% must go to investment in eleven priority areas and the remainder is consumed. The 30-50% of benchmark revenue not allocated to the annual budget is put in sovereign wealth funds which invest in foreign assets; a minimum of 30% must go to a heritage or intergenerational fund and the rest to a stabilization or liquidity fund.

Figure 2.3: Reserves, extraction rates and oil rents for Ghana.

(a) Reserves and extraction rates (b) Resource rents



Projected oil reserves levels (panel a, left axis), extraction rates (panel a, right axis) (both not in efficiency units) and corresponding resource rents as a share of non-resource GDP (panel b) for Ghana. See §2.4.3 for details.

production profile until 2029. We adopt a more optimistic estimate of 1 billion barrels of oil reserves at the end of 2011 and assume a linear increase of production from 30 in 2012 to 45 billion barrels of oil per year in 2014 followed by constant production at that level until 2017. Thereafter, we consider two scenarios in Figure 2.3: (A) continue at 2017 level until exhaustion in 2034; (B) linear decline until exhaustion in 2052. Based on the estimated extraction costs for Norway of \$4-14 per b.o.e., part of which are offshore, we guess extraction costs to be rather higher at \$25 per b.o.e., reflecting the fact that the reserves are offshore and in deep water. At peak production in 2014, annual resource rents are approximately \$3.8 billion, \$140 per citizen (not in efficiency units) or 0.5% of non-resource GDP and declining rapidly thereafter.

2.5 Calibration

Before we can calculate our estimates of the optimal intergenerational and liquidity funds for each of the three countries discussed in §2.4 and the optimal investing to invest strategy for Ghana which suffers from capital scarcity, we present the calibration we use for our calculations in Table 2.1. The assumptions regarding future extraction costs have already been discussed in §2.4. Our estimates of the optimal size of the intergenerational and liquidity funds apply to the economy as a whole as we do not distinguish between private and public oil rents.

Table 2.1: Calibrations details.

	Norway	Iraq	Ghana
Future extraction costs Ψ	15 \$/b.o.e.	10 \$/b.o.e.	25 \$/b.o.e.
Population growth rate n	0.5%/year	2.3%/year	2.3%/year
Rate of technical progress g	1.0%/year	1.8%/year	2.5%/year
Real risk-free interest rate r^*	3.4%/year	6.0%/year	7.0%/year
Growth-corrected rate of time preference ρ	1.9%/year	1.9%/year	2.2%/year
Processes for the world oil price:			
(i) random walk	drift $\upsilon_P = 0$ and volatility $\sigma_P = 0.40$ year$^{-1/2}$ (natural gas: $\upsilon_P = 0$, $\sigma_P = 0.21$ year$^{-1/2}$)		
(ii) AR(1) process	mean price $\exp(m_P) = 110$ \$/barrel, mean reversion coefficient $\eta_P = 0.06$ year^{-1} and volatility $\sigma_P = 0.26$ year$^{-1/2}$ (natural gas: $\exp(m_P) = 32$ \$/b.o.e., $\eta_P = 0.06$ year^{-1} and $\sigma_P = 0.20$ year$^{-1/2}$)		
Investing to invest assumptions for Ghana	Interest spread $\Pi(B) = 10^{-4}\exp(6.294)(\exp(-1.9B/31.3)-1)$ with indebtedness $-B$ measured in efficiency units		
	Share of private capital in value added $\alpha = 0.1$		
	Output elasticity of public capital $\beta = 0.17$		
	Depreciation of public capital $\delta^* = 0.025$ year^{-1}		
	Adjustment cost parameter for public capital $\phi = 34.5$ year		
	Initial public capital $S_0 = \$10.1$ billion		
	Total factor productivity $A_0 = \$24.9$ billion		

2.5.1 Population growth and technical progress

For Norway, we use a population growth rate of $n = 0.5\%$ per year[21] and a rate of technical progress of $g = 1\%$ per year, which implies a trend growth rate of 1.5% per year. For Iraq, we use a population growth rate of $n = 2.3\%$ per year. Iraq's growth rate has been quite volatile. We use Iraq's average growth rate during 2005-2011 for the trend rate of real GDP growth of 4.1% per year. We thus set the rate of technical progress to $g = 1.8\%$ per year. Ghana's population growth has been relatively stable at between 2.3% and 2.4% per year during 2001-2012 (from World Bank Development Indicators), and we set $n = 2.3\%$ per year. Ghana's productivity growth rate has averaged 3% per year during 1991-2001 and 4.4% per year during 2001-2011 (from World Bank Development Indicators). Recently, real GDP growth for Ghana has risen sharply from 4.1% per year in 2009 to 13.6% per year in 2011, reflecting the start of oil production in 2010. We take a more modest but still large annual trend rate of growth for the long horizon under consideration of $n + g = 4.8\%$ per year, where $g = 2.5\%$ per year.

2.5.2 Stochastic dynamics of the oil price

To calculate projected windfalls, forecasts are needed for the world oil price. We thus estimate a time-series model for the world oil price, which is shown in Fig-

[21] From Norway's official long-term forecast `http://www.ssb.no/folkfram_en/main.html`.

ure 2.4.[22] Evidence suggests that it is difficult to reject the hypothesis that the log of the real oil price follows a random walk without drift (Hamilton 2009). We thus first estimate a random-walk process. For the period 1960-2011, we obtain the Maximum Likelihood (ML) estimates of the drift and volatility parameters in (2.3.6): $\hat{\upsilon}_P = 0.087$ year^{-1} and $\hat{\sigma}_P = 0.40$ year$^{-1/2}$, where the hats denotes our best estimates. Since $\hat{\upsilon}_P$ is statistically insignificant (t-ratio = 1.55), we set $\upsilon_P = 0$ and thus ignore long-run trends in the oil price.

For Norway, approximately half of resource revenues come from natural gas.[23] Figure 2.4 shows significant correlation between the oil and gas price and comparable, yet smaller, volatility. We obtain the following ML estimates of the drift and volatility parameters in (2.3.6) for the real price of natural gas: $\hat{\upsilon}_P = 0.048$ year^{-1} and $\hat{\sigma}_P = 0.21$ year$^{-1/2}$ with the drift again insignificant (t-ratio = 1.67).

Since the size of precautionary saving is quite sensitive to even small degrees of mean reversion (cf. Bems and de Carvalho Filho (2011)), we also estimate the AR(1) process (2.3.8), but without trend, and obtain for crude oil $\hat{m}_P^* \hat{\eta}_P = 0.27$ year^{-1} (t-ratio = 1.33) and $\hat{\eta}_P = 0.066$ year^{-1} (t-ratio = 1.16) and $\hat{\sigma}_P^* = 0.29$ year$^{-1/2}$, which corresponds to a mean price of $\exp(\hat{m}_P) = 110$ \$/barrel and a volatility of $\hat{\sigma}_P = 0.26$ year$^{-1/2}$. For the gas price over the same period 1960-2011, we obtain $\hat{m}_P^* \hat{\eta}_P^* = 0.21$ year^{-1} (t-ratio = 1.70) and $\hat{\eta}_P^* = 0.066$ year^{-1} (t-ratio = 1.54), which corresponds to a mean price of $\exp(\hat{m}_P) = 32$ \$/b.o.e., $\hat{\eta}_P = 0.064$ year^{-1} and a volatility of $\hat{\sigma}_P = 0.20$ year$^{-1/2}$.

A random walk process has drawbacks (e.g. uncertainty that grows monotonically with time and the unlimited persistence of current shocks into the future). Yet, so does the AR(1)-process with the estimated coefficients being very sensitive to the sample period and the inclusion of a trend. To overcome this, we use both the estimated random walk and AR(1) processes for the oil and natural gas price and conduct a thorough sensitivity analysis.

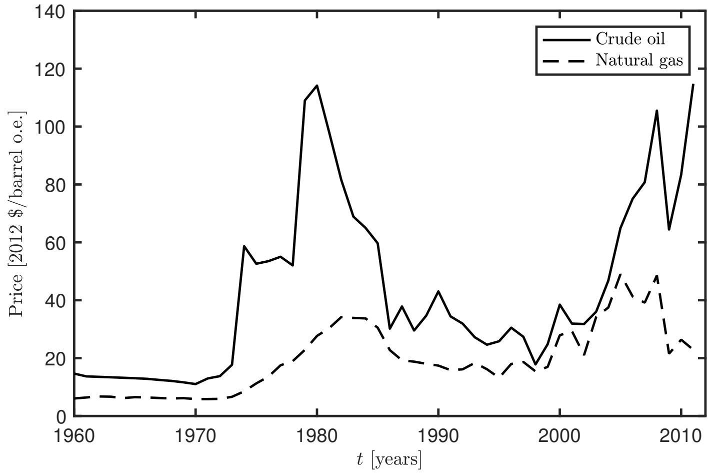
2.5.3 Interest rates and interest premium on national debt

Although the real rate of return on the Norwegian SWF has been 2.4%/year on average during the period of its existence, 1998-2011 (NBIM 2011), we use the larger historical average of the real interest rate on short-term Norwegian govern-

[22] We use a historical annual series for the price of crude oil (BP 2012), expressed in 2012 prices using CPI data for the United States (from OECD Economic Outlook No. 91). As all figures are in U.S. dollars to compare between countries, we use U.S. inflation data to obtain the real oil price and, in doing so, we abstract from purchase power parity considerations.

[23] We use the UK natural gas price for the period of 1996-2011 (BP 2012), which seems the best available location. In the absence of generally available data for Europe for the period 1971-1995, we use the U.S. Natural Gas Wellhead Price for that period, available online: `http://www.eia.gov/dnav/ng/hist/n9190us3a.htm` (Washington, D.C., U.S. Energy Information Administration)

Figure 2.4: Real crude oil and natural gas prices 1960-2011.



Historical behaviour of the crude oil and natural gas price for the period 1960-2011 used for calibration. See §2.5.2 for details.

ment bonds during 1970-2011 of 3.4%/year as proxy for the real risk-free rate r^*.[24][25] This yields a growth-corrected interest rate r of 1.9%/year. For the benchmark, we ensure that growth in consumption in efficiency units is zero, so $\rho = r = 1.9\%$/year. For Iraq, we assume $r^* = 6\%$/year, so that the growth-corrected interest rate r is 1.9%/year. We also set ρ equal to 1.9%/year. For Ghana we set $r^* = 7\%$/year, so the growth-corrected interest rate r is 2.2%/year and we set $\rho = 2.2\%$/year accordingly ($\rho = r$).[26]

Our model assumes that countries with a high debt or capital scarcity pay an interest premium on their foreign debt. We assume this is the case for Ghana, but not for Norway or Iraq. From cross-country regressions (van der Ploeg and Venables 2011) obtain the interest spread schedule $\Pi(B) = 10^{-4}\exp(6.294)\times [\exp(-1.9B/31.3) - 1]$, where 6.294 is the mean log of the spread. The initial value of $-B$ is a debt-GDP ratio and from then on it is adjusted by trend growth in GDP. This implies that a 10%-point increase in the debt-GDP ratio raises the interest differential by 6.9%-points if the economy has an initial debt-GDP ratio of 100 percent (or 1.3%-points if it has zero foreign debt). There is no empirical

[24] We do not consider the effect of risky investments until Chapter 4

[25] Excluding recent crisis years when rates in some safe countries have plummeted to below zero, we would have obtained 3.7%/year. We do not take the 2.4%/year of the Norwegian SWF as this may be low due to the dominant effect of the downturn of 2009. Bems and de Carvalho Filho (2011) also use a high rate of return of 4%/year.

[26] Hence, $\rho^* = r^* - \eta g = 1.4\%$/year, 2.4%/year and 2.0%/year for Norway, Iraq and Ghana, respectively if $\eta = 2.0$.

support for an effect of the size of the windfall on the interest premium.[27]

2.5.4 Public investment: productivity and inefficiency

Recent survey evidence suggests that only 40 to 60 percent of spending on public investment realizes effective accumulation of public-sector capital (Dabla-Norris et al. 2011, Gupta et al. 2011). As public investment is increased, its efficiency deteriorates (Berg et al. 2011, van der Ploeg 2012). To capture that absorption constraints frustrate rapid economic development and to allow for a more realistic calibration of the model, we include internal costs of adjustment. The ratio of investment that delivers public capital to total investment spending is the 'public investment measure of inefficiency' (*PIMI*). It is given by $PIMI = 1/(1 + \phi I/(2S))$, so that ramping up public investment lowers the *PIMI*. A high value of ϕ reflects absorption constraints: higher marginal returns on public capital are required. Investment is more inefficient in the early stages of economic development when public investment rates are high. Supposing that only 40% of spending on public investment is effective, the steady-state *PIMI* equals $1/(1 + \phi\delta/2) = 0.4$, but in the early stages of development and during the windfall less of investment outlays is actually delivered (the *PIMI* is low) as public investment rates (I/S) are higher.

A ballpark estimate for the output elasticity with respect to the stock of public capital is 0.15 (Bom and Ligthart 2010). In line with this evidence, we set $\beta = 0.15/(1-\alpha)$. If we set the share of private capital in aggregate production equal to $\alpha = 0.1$ (not unreasonable for a country like Ghana), we obtain a reduced-form output elasticity of $\beta = 0.167$. Hence, doubling the stock of public capital boosts output by roughly 17 percent. Since we use Ghana to illustrate how capital scarcity affects how the windfall is used for investing to invest, we calibrate our model very coarsely to Ghanaian data. We set the depreciation rate of public capital to $\delta^* = 0.025$ year^{-1}, which corresponds to an expected lifetime of 40 years, so that $\delta = \delta^* + n + g = 0.087$ year^{-1} for Ghana. The adjustment cost parameter for public investment follows from the steady-state expression for the *PIMI*: $\phi = 3/\delta = 34.5$ years. We set the initial stock of public capital to half its steady-state value, $S(0) = 0.5\left[((1-\alpha)\beta A_0)/(r + \delta + \phi\delta(r + 0.5\delta))\right]^{\frac{1}{1-\beta}}$ (using (2.3.15)), and calibrate A_0 to match Ghana's non-resource GDP in 2012, $\beta A_0 S^\beta$ = \$36.7 billion/year. This gives $S(0)$ =\$10.1 billion/year and A_0 = \$24.9 billion/year for Ghana.

[27]The coefficients for the effects of the public and publicly guaranteed debt to GNI, the ratio of central bank reserves to GDP and the probability of default on the log of the interest rate spread are, respectively, 1.89, − 4.14 and 0.296 (van der Ploeg and Venables 2011).

2.6 Estimates of intergenerational and liquidity funds

2.6.1 Norway

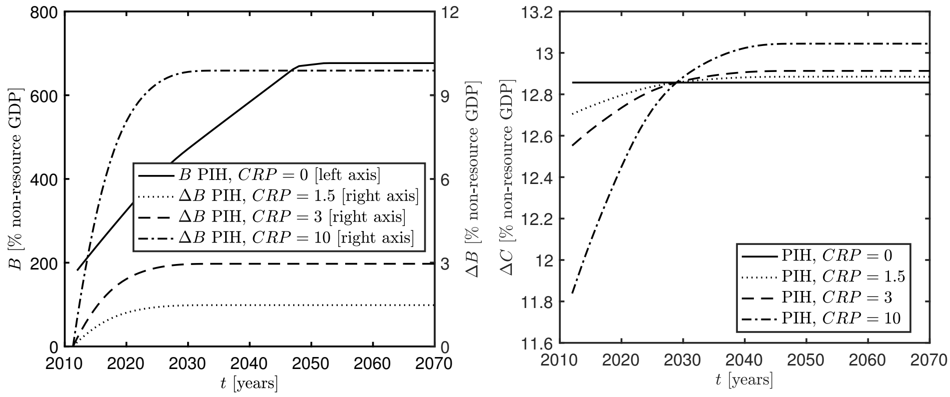
To determine the size of the optimal intergenerational buffer, we calculate the optimal savings responses when oil price volatility is set to zero or prudence is absent ($CRP = 0$). The continuous line in Figure 2.5a shows that this eventually results in a gradual sovereign wealth accumulation, from 1.8 to 6.8 times non-resource GDP (see left axis). This saving response permits a permanent increase in consumption of 12.9% of non-resource GDP. This amounts to an annual annuity of $8.5k for each Norwegian citizen in 2012 (and growing at the trend rate of $g = 1.0\%$ per year). Having set volatility to zero, this buffer shows the optimal development of the intergenerational fund.

To obtain an estimate of the order of magnitude of the optimal size of the liquidity fund, we also calculate the optimal savings response with oil price volatility described by the AR(1)-estimate of the stochastic dynamics of the world oil price (2.3.8). Taking a ballpark value for the coefficient of relative prudence of 3, the dashed line in 2.5a (right axis) indicates the prudent saving response. Norway should thus accumulate an additional 3% of non-resource GDP in its liquidity fund, which does not appear large relative to the size of the intergenerational fund. However, it still corresponds to an additional fund of $2k per Norwegian citizen in 2012 growing at a rate of 1.0% per year. If the coefficient of relative prudence is 10, the dot-dashed line in the left panel (right axis) indicates that it is optimal to have a liquidity fund of 10% of non-resource GDP, which still amounts to only about one seventieth of the intergenerational fund. With a prudence coefficient at the lower end of the plausible range of 1.5, the dotted line shows that the optimal liquidity fund is only 1.5% of non-resource GDP or 0.2% of the intergenerational fund.

Figure 2.5b shows the corresponding consumption increments. Taking account of oil price volatility and using the ballpark coefficient of relative prudence of 3, consumption increases initially by only 12.6% and then rises to 12.9% of non-resource GDP. The prudent saving response thus implies less consumption today (−0.3% of non-resource GDP with $CRP = 3$) and more consumption in the long run (+0.06% with $CRP = 3$). These prudential tilts in the consumption profile are greater if relative prudence is greater, begging the question what the socially optimal degree of prudence is. For $CRP = 10$ (dot-dashed line), precautionary savings are 1% of non-resource GDP in 2012, resulting in a sustained increase in consumption of 0.2% of non-resource GDP after the windfall. The temporary nature of Norway's windfall implies that the marginal propensity to consume with respect to the price of a barrel of oil (2.3.20) falls monotonically with time. Therefore, the prudence effect also falls with time, and the upward tilt of the path for the consumption increment reduces with time accordingly.

Figure 2.5: Optimal savings and consumption responses to Norwegian windfall (scenario A).

(a) Fund build-up (b) Consumption increment



Norway's optimal future intergenerational fund build-up (left panel, left axis), its liquidity fund build-up (left panel, right axis) and the corresponding incremental consumption path compared to the no windfall case (all as a percentage of non-resource GDP). Three different degree of relative prudence ($CRP = 1.5, 3, 10$) are considered for the liquidity fund. $CRP = 0$ corresponds to the intergenerational fund or the case in which the decision-maker treats the windfall as deterministic. See §2.6.1.

Table 2.2: Sensitivity results for Norway (percent of non-resource GDP).

	$\Delta C^I(t)$	$-\Delta C^L(2012)$	$\Delta C^L(\infty)$	$A^I(\infty)$	$A^L(\infty)$
Base case	13	0.3	0.06	6.8×10^2	3.0
Extraction scenario B	13	0.4	0.07	6.7×10^2	3.9
Initial reserves +50%	14	0.4	0.11	7.6×10^2	5.7
$r^* = 2.4\%$/year	6.7	0.09	0.01	7.5×10^2	0.9
$g = 0.5\%$/ year	16	0.5	0.12	6.8×10^2	5.1
Random walk	14	1.3	0.20	7.2×10^2	11
AR(1) with $\eta_P = 0.09$ year^{-1}	13	0.2	0.04	6.7×10^2	2.0
Lower mean oil and gas price ($\exp(m_P) = 70$ and 20.4 \$/b.o.e.)	11	0.2	0.03	5.6×10^2	1.6

Sensitivity analysis for Norway

The benchmark presented in Figure 2.5 has initial reserves of 46.6 billion b.o.e. with extraction scenario A, $\eta = 2$, $n = 0.5\%$/year, $g = 1\%$/year, $r^* = 3.4\%$/year, for crude oil $\exp(\hat{m}_P) = 110$ \$/barrel, $\hat{\eta}_P = 0.06$ year^{-1}, $\sigma_P = 0.26$ year$^{-1/2}$ and zero trend, and for natural gas $\exp(m_P) = 32$ \$/b.o.e., $\eta_P = 0.06$ year^{-1}, $\sigma_P = 0.20$ year$^{-1/2}$ and zero trend in (cf. (2.3.8)). Initial price levels in 2011 were 110 \$/b.o.e for oil and 51 \$/b.o.e. for gas. Table 2.2 shows the results of a sensitivity analysis around this benchmark.

If linearly declining extraction rates are used from 2030 (scenario B), the windfall is spread out over a longer period, which results in a somewhat smaller intergenerational fund and thus a smaller permanent increase in consumption. Thanks to discounting, the effect is small. On the other hand, a less rapidly declining oil depletion rate induces a bigger liquidity fund and thus consumption rises by less initially and rises by a little more in the long run (compared to the no-windfall outcome). Since initial oil and gas reserves are highly uncertain, Table 2.2 also report what happens if reserve levels are at the higher end of the estimated range. Evidently, both the intergenerational and the liquidity funds and the resulting consumption increments are larger but, due to discounting, less than proportionally so.

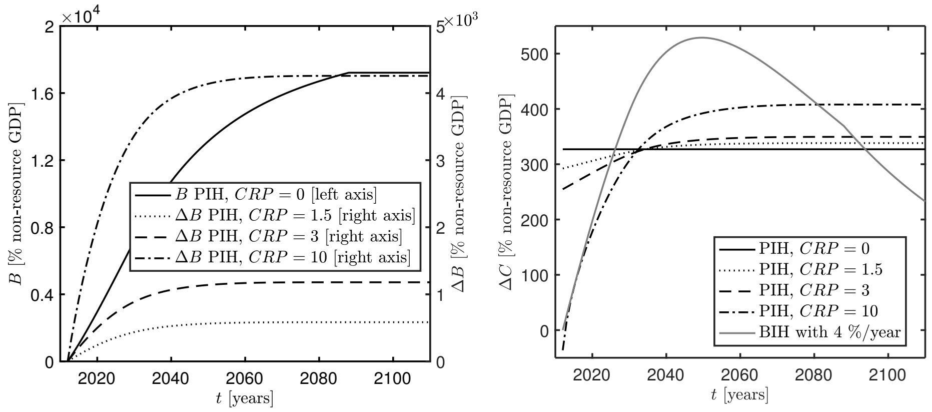
A lower real return on sovereign wealth has dramatic effects. Although it boosts the intergenerational fund by more than 70% of non-resource GDP, the resulting permanent increase in consumption is only 6.7% instead of 12.9% of non-resource GDP. As is clear from equation (2.3.18), the marginal propensity to consume out of an oil price shock is much less with a smaller real return, and thus from (2.3.16) we see that the prudence effect and the liquidity fund that is accumulated are much smaller. A lower economic growth rate boosts oil production in efficiency units and thus from (2.3.20) boosts the marginal propensity to consume out of an oil price shock and from (2.3.16) induces a much bigger liquidity buffer, i.e. 5.1% instead of 3.0% of non-resource GDP. Although the size of the intergenerational fund is unaffected, the increase in the growth-corrected real return induces a much bigger permanent increase in consumption.

If our estimates of a random walk for the oil price are used instead of an AR(1) process, we find a bigger prudence effect as shocks now persist forever (see (2.3.18) and (2.3.20)), and thus the optimal size of the liquidity fund is a factor 3.5 bigger. The precautionary consumption tilt is larger too. With a bigger mean-reversion parameter of 0.09 year^{-1}, the results go the opposite way: the prudence effect is small and thus the optimal size of the liquidity fund is smaller. This illustrates the large sensitivity of the optimal size of the liquidity to the degree of mean reversion in the oil price, which is notoriously difficult to pin down statistically with any great confidence.

Figure 2.6: Optimal savings and consumption responses to Iraqi windfall (scenario A).

(a) Fund build-up

(b) Consumption increment



Iraq's optimal future intergenerational fund build-up (panel a, left axis), liquidity fund build-up (panel a, right axis) and corresponding incremental consumption path compared to the no windfall case (all as a percentage of non-resource GDP). Three different degree of relative prudence ($CRP = 1.5, 3, 10$) are considered for the liquidity fund. $CRP = 0$ corresponds to the intergenerational fund or the case in which the decision-maker treats the windfall as deterministic. The bird-in-hand rule (BIH) with an annual consumption of 4%/year of accumulated assets initially closely matches the consumption path of a very prudent ($CRP = 10$). See §2.6.2.

Finally, a lower mean price for oil and gas ($\exp(m_P) = 70$ \$/barrel for crude oil and $\exp(m_P) = 20.4$ \$/b.o.e. for natural gas, i.e. a little more than a third less for both) scales down the optimal size of the intergenerational fund less than proportionally and scales down the liquidity fund more than proportionally (as initial prices are unchanged). Hence, the sustained increase in consumption rises less than proportionally and the precautionary tilt in the consumption path increases more than proportionally.

2.6.2 Dominance of Iraq's liquidity buffer

Not taking account of oil price volatility, Iraq should accumulate sovereign wealth amounting to a colossal 172 times non-resource GDP, reflecting both the huge size of its windfall and its low level of non-resource GDP. This ensures a sustained annual consumption increment of 3.3 times non-resource GDP. This corresponds to an ever-lasting annual annuity of \$1.5k for each Iraqi citizen in 2012, which will grow from then on at 1.8% per year. Although the Norwegian windfall is much smaller than that of Iraq, relative to its population the Norwegian windfall is larger, and therefore Iraq's annuity for each citizen is smaller than for Norway (\$8.5k).

Since the Iraqi windfall is so large and lasts so long, the marginal propensity

Table 2.3: Sensitivity results for Iraq (percent of non-resource GDP).

	$\Delta C^I(t)$	$-\Delta C^L(2012)$	$\Delta C^L(\infty)$	$A^I(\infty)$	$A^L(\infty)$
Base case	3.3×10^2	7.2×10^1	2.2×10^1	1.7×10^4	1.2×10^3
Extraction scenario B	3.2×10^2	7.3×10^1	2.2×10^1	1.7×10^4	1.2×10^3
Initial reserves +50%	3.3×10^2	7.3×10^1	2.3×10^1	1.7×10^4	1.2×10^3
r^* = 5%/year (-1%/year)	1.9×10^2	3.2×10^1	4.7	2.1×10^4	5.2×10^2
g = 0.8%/year (-1%/year)	5.0×10^2	1.4×10^2	8.2×10^1	1.7×10^4	2.8×10^3
Random walk	3.3×10^2	4.3×10^2	1.5×10^2	1.7×10^4	9.0×10^3
AR(1) with η_P = 0.09 year^{-1}	3.3×10^2	4.5×10^1	1.4×10^1	1.7×10^3	7.6×10^2
Lower mean oil price ($\exp(m_P)$ = 70 \$/barrel)	2.5×10^2	4.9×10^1	1.4×10^1	1.3×10^4	7.1×10^2

to consume out of oil wealth (2.3.18) is relatively large, and thus the prudence effect shown in (2.3.17) is very large. Iraq is thus very vulnerable to oil price volatility and needs to build up a relatively large volatility buffer or stabilization fund compared to its generational fund. In fact, for CRP = 10 (see dot-dashed line in Figure 2.6) the consumption increment in 2012 is negative. This does not happen for lower degrees of prudence such as CRP equal to 1.5 or 3 (dotted and dot-dashed lines in Figure 2.6, respectively). The prudent gradual accumulation of financial assets over eight decades leads in the base case with CRP = 3 to a volatility buffer of 11.8 times non-resource GDP ($5.5k per Iraqi), which amounts to 6.9% of the intergenerational fund. This brings the total sovereign wealth fund up to 184 times non-resource GDP or $86k per Iraqi citizen, with the latter growing at 1.8% per year. To achieve this amount of prudent saving, consumption has to fall compared with the zero-prudence outcome, so that it rises by 255% instead of 327% of non-resource GDP initially (i.e. by $0.3k per citizen less) and then rises to 349% of non-resource GDP in the long run. Interestingly, the consumption path for the initial years for very high degrees of prudence is not that different from the BIH path.

Sensitivity analysis for Iraq

The base case reported in Figure 2.6 used n = 2.3%/year, g = 1.8%/year, r^* = 6%/year, extraction scenario A, and the same estimated oil price process (2.3.8) used for Norway. Table 2.3 offers some sensitivity results. Whether extraction scenario A or B is used does not significantly affect the size of the intergenerational and liquidity funds nor the consumption increments. Due to the high discount rate (r^* = 6.2%) and long time horizons involved for the Iraqi windfall, additional oil reserves do not have much effect either.

More disappointing growth prospects (g = 0.8%/year) do not affect the final size of the intergenerational fund (as consumption in efficiency units is smoothed across generations), but does lead to a bigger permanent consumption increment.

Table 2.4: Sensitivity results for Ghana (percent of non-resource GDP).

	$\Delta C^I(t)$	$-\Delta C^L(2012)$	$\Delta C^L(\infty)$	$A^I(\infty)$	$A^L(\infty)$
Base case	2.5	0.03	0.004	1.2×10^2	0.2
Extraction scenario B	2.4	0.03	0.004	1.1×10^2	0.2
Initial reserves +50%	2.9	0.04	0.006	1.3×10^2	0.3
g = 1.5%/year (-1%/year)	3.7	0.06	0.01	1.2×10^2	0.4
Random walk	2.5	0.06	0.008	1.2×10^2	0.4
AR(1) with $\eta_P = 0.09$ year^{-1}	2.5	0.02	0.003	1.2×10^2	0.1
Lower mean oil price ($\exp(m_P) = 70$ \$/barrel)	2.0	0.02	0.002	9.3×10^1	0.1

It also leads to a significantly larger liquidity fund and precautionary tilt of the consumption profile underlining the reducing effect of productivity growth on the need for precautionary saving.

A random walk for the world oil price does not affect the size of the optimal intergenerational fund, but leads to liquidity fund a factor 8 larger. More mean reversion in the oil price leads to liquidity fund about a third smaller. Finally, a lower mean oil price ($\exp(m_P)$ = \$70 instead of \$110 per barrel) leads to a less than proportional fall in the intergenerational fund and to a more than proportional fall in the liquidity fund.

2.6.3 Small intergenerational and liquidity buffers for Ghana

Our estimates of the optimal size of the intergenerational and liquidity funds are reported in Table 2.4. For the base case with n = 2.3%/year, g = 2.5%/year, t^* = 7%/year and extraction scenario A, we find that the optimal size of the intergenerational fund in the long run is 115% of non-resource GDP, which permits a sustained increase in consumption of 2.5% of non-resource GDP. This amounts to an annuity of a mere 37 \$ per citizen, albeit growing in real terms at 2.5% per year. The optimal size of the liquidity fund for Ghana and the resulting precautionary tilt of the consumption time path are hardly noticeable. Qualitatively, the same insight is obtained for extraction scenario B, higher oil reserves and a random walk process for the world oil price or more mean reversion in the process for the world price as for the two other countries. A lower growth rate of the economy leads to a less rapid decline of oil rents in efficiency units and to a larger, albeit still hardly significant, liquidity fund. Since the windfall is relatively small and temporary, (2.3.17) and (2.3.18) indicate that the prudence effect is much smaller compared to Norway and even smaller compared to Iraq.

2.7 Capital strategy and investing to invest strategy for Ghana

Given that Ghana is likely to suffer from capital scarcity, it should allocate (in contrast to Norway) part of its windfall not to sovereign wealth but to investment in the domestic economy. The level of present consumption should then be below that of future consumption for two reasons. First, if there is capital scarcity, it is optimal to pay off debt and reduce the interest burden. This effect is especially strong if intergenerational inequality aversion is not so large and capital scarcity is substantial. Second, the need to build a precautionary saving buffer tilts consumption towards the present. We have seen that given the small and temporary nature of Ghana's windfall the second effect is tiny. Hence, we abstract from oil price volatility and focus on the trade-off between capital scarcity and the need to invest versus the desire to transfer wealth towards future generations.

The calibration is discussed in §2.5.4 and §2.5.3.[28] The coefficients chosen imply the following steady state for Ghana: $S(\infty)$ = \$20.2, $Y(\infty)$ = \$41.1 and $C\infty)$ = \$23.6 billion (2012 prices). We set Ghana's initial external stock of public and publicly guaranteed external debt for 2012 equal to $-B_0$ = \$6 billion (14% of GDP). The consumption increment that can be sustained under the permanent income rule is \$0.93 billion/year (2012 prices) in 2012, growing at 4.8% per year from then on. Figure 2.7 portrays the optimal development paths for Ghana without windfall (dashed lines) and with windfall (solid lines) for variables measured in efficiency units.[29]

Without the windfall Ghana will grow from a sub-optimally low level of its public-sector capital stock along its development path. The gradual rise in public capital and output is associated with a rising path for public investment. Because of the absorption constraints the economy faces in the early periods of development when investment is high, the efficiency of public investment gradually improves with time (witness the rise in the *PIMI*). In the very long run output grows from \$36.6 to \$41.1 billion whilst the public and publicly-guaranteed debt vanishes.

The effects of the oil windfall are to allow a more rapid build-up of public investment (signalled by a temporary higher social value of public investment, q) and a speeding up of the process of economic development. This inevitably leads to a temporary deterioration of the efficiency of public investment (lower *PIMI*). For a considerable amount of time the stock of public capital in efficiency units is higher than without the windfall which leads to long-lasting higher levels of output

[28] We approximate oil rents in efficiency units by $N(t) = 3.8\exp[-0.068(t-2012)]$, so $N(t=0)$ = \$3.8 billion and oil rents decline in 2012 dollars at 2.0%/year (as $n+g$ = 4.8%/year). This matches 2012 in-situ oil wealth of \$42 billion and the annuity value of oil wealth in efficiency units as 2.2 percent of this, i.e. $NP(t=0)$ = \$0.93 billion/year.

[29] The optimal policy simulations are obtained from a linearization of the state-space model.

in efficiency units. Given that Ghana's windfall is fairly short and small, this is a modest but significant increase. Still, the windfall allows for an increase in consumption from \$21.6 to \$24.0 billion at the start of the windfall and from \$23.0 to \$24.2 billion in efficiency units (\$49.6 to \$52.2 billion unadjusted) in 2028. The windfall allows external public and publicly-guaranteed debt to be paid off more rapidly, so that the social cost of borrowing falls more rapidly and public investment is stimulated. Net government assets (value of public capital minus public and publicly-guaranteed debt) given by $qS + B$, where q is the "social price" of public capital, jump up from \$44.6 to \$46.6 billion on impact due to the jump increase in the social value of public capital q.[30] Afterwards, net assets continue to grow initially more rapidly with the windfall than without the windfall but eventually less rapidly so.

There are two key insights. First, much of the consumption increment comes in the early periods of the windfall. This reflects that future generations will be richer so more of the windfall is spent on current generations to avoid too much intergenerational inequality (more so if σ is smaller). Second, a substantial part of the windfall is spent on public investment which reflects that capital scarcity means that the economy has a sub-optimally low level of public investment. The windfall must thus be used to fund an investing-to-invest strategy, too boost consumption and to pay off foreign debt in a balanced way.

Partial-equilibrium application of permanent-income (PIH) and bird in hand (BIH) rules do not affect capital formation and output. Compared with the no-windfall paths, the PIH rule leads to a permanent increase in consumption of \$0.93 billion and a long-run size of the intergenerational fund of \$42.2 billion (both in efficiency units). The BIH rule with putting all windfall revenue into a fund and extracting 4 percent of this fund each year to fund the general budget leads to a temporary build-up of more than \$9 billion. These rules do little to stimulate the economy and thus lead to much lower consumption in the next three or four decades than the optimal 'investing to invest' trajectories.

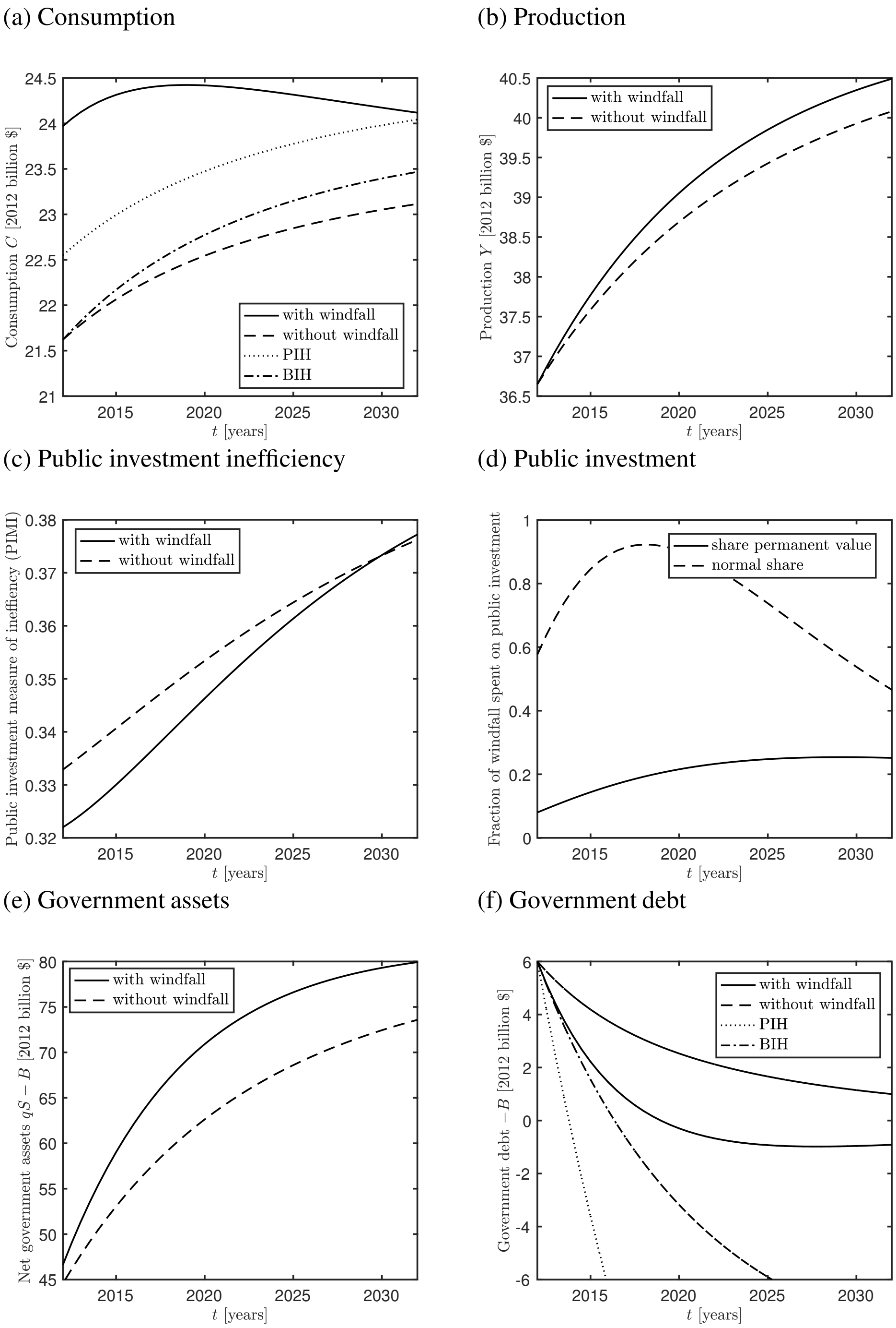
Although our calibration is ad hoc with data on many variables hard to come by, the qualitative insight from Figure 2.7 that the 'investing to invest' strategy is better for developing economies with capital scarcity than the naive PIH and BIH rules remains valid.

2.8 Conclusions

This chapter has highlighted the different roles that intergenerational, liquidity and investment funds can play in managing and harnessing oil windfalls. An intergen-

[30]Note that the public investment rate, $I/S = 2(1 - PIMI)/PIMI$, and the social price of public capital, $q = 1 + \phi I/S$, are negatively related to the $PIMI$.

Figure 2.7: Harnessing Ghana's windfall for domestic investment.



Ghana's optimal future response to a windfall taking account capital scarcity comparing the case with and without windfall: consumption also showing the permanent income hypothesis (PIH) (a); the non-resource output (b); the public investment measure of inefficiency (*PIMI*) (c); the share of the windfall spent on public investment compared to the share of the permanent value of the windfall (d); total net government assets (public capital minus government debt) (e); and government debt also showing the permanent income hypothesis (PIH) and the bird in hand rule (BIH) (f). See §2.7. Y, C, D and $qS + B$ are in billions of 2012 dollars (in efficiency units).

erational fund is needed to smooth the benefits of time-varying windfalls across generations. A liquidity fund is required to protect the economy against stochastic oil price volatility in addition to the usual arguments in favour of more political stability and flexibility of the economy. A liquidity fund offers an important alternative for hedging against oil price volatility, since hedging and related structured products have too many economic costs and political risks. Furthermore, markets are too thin to deal with size and duration of a windfall of a country such as Iraq. The size of the liquidity fund should be larger if oil income volatility is higher, oil income makes up a larger share of total income, and governments are more prudent. More notably, its size also depends on the marginal propensity to consume out of a windfall and thus strongly on the interest rate, the duration of the windfall and the degree of mean reversion: only if oil price shocks lead to consumption shocks, do they necessitate precautionary buffers. If the windfall is temporary, the oil rents are largely saved, and little precautionary saving is needed (Ghana). If the windfall is permanent combined with the almost random walk behaviour of the oil price, shocks in the oil price lead directly to shocks in consumption and large precautionary buffers are required (Iraq).

Table 2.5 sums up our base case estimates of the optimal sizes of the intergenerational and liquidity funds and the corresponding consumption increments. Iraq's windfall is the largest and lasts longest, so its liquidity fund is much larger relatively: 6.9% of its intergenerational fund for Iraq, whilst for Norway it is only 0.4% and for Ghana 0.2%. This reflects the fact that Iraq's windfall lasts longer than that of Norway and longer still than that of Ghana and, more importantly, that the Iraqi windfall makes up an enormous share of total consumption and will continue to do so for the foreseeable future. Ghana and Iraq thus have relatively a larger intergenerational fund. Since Iraq has a much larger population (33 million) than Norway (5 million), its oil annuity per citizen is less despite having a bigger windfall in absolute terms. Ghana's windfall does not really necessitate a liquidity fund.

By setting $\rho = r$, we have effectively assumed that future growth cannot be used as collateral for borrowing on the international capital market and we thus smooth consumption in efficiency units, not consumption itself. Productivity growth of non-resource income then acts as a natural mechanism that curbs the need for precautionary saving. For Iraq, our assumptions imply that in 2012 approximately half of the windfall ($(670 - 327)/670 \approx 51\%$), a staggering 670% of non-resource GDP, should be saved in the intergenerational fund, 11% in the liquidity fund ($72/670 \approx 11\%$) and the remaining 38% should be consumed (cf. Table 2.5). Accordingly, the intergenerational fund ultimately reaches a mind-blowing 172 times non-resource GDP, which evidently calls into doubt our assumption that $\rho = r$, yet corresponds to reasonable consumption increments in 2012.

An alternative is to allow future growth to act as collateral in which case one would smooth per-capita consumption across generations (by setting $\rho^* = r^*$). Assets per capita will then vanish asymptotically as non-resource income grows forever. If the windfall is big enough, the majority of the windfall is consumed and only a small part may still be saved in an intergenerational fund.[31] As planned consumption no longer grows, a larger liquidity fund is needed.[32]

Out of a sample of 31 oil producers 21 have funds of which 10 focus on stabilization and 8 on stabilization and saving (IMF, 2005). In practice, the size of the liquidity fund should depend on the features highlighted in our analysis, but also on the costs of volatility to the domestic economy or the opportunities for borrowing in the downturn. Stabilization or liquidity funds are typically contingent on the oil price or oil revenue. In the real world, the political risk of such funds being looted also matters. If this is a serious risk, government will have a bias towards partisan, illiquid investment projects at the expense of saving in liquid sovereign assets and/or growth-enhancing (neutral) investment projects.

For developed countries with good access to world capital markets, not even part of the windfall should be spent on domestic investment. However, for countries with capital scarcity it makes sense to channel their windfalls of foreign exchange into a domestic investment fund if the expected return on domestic investment and the cost of borrowing are more than the return on sovereign wealth. Not all developing economies can absorb the extra windfall-induced demand for non-traded consumption and investment goods, especially if absorptive capacity is limited. In that case, there is a rationale for a parking fund in addition to an intergenerational and a liquidity fund.

Our illustrative calculations indicate that the optimal liquidity buffer for Ghana is very small relative to its intergenerational fund even for very high degrees of prudence, Norway's optimal liquidity fund is small and Iraq should have a much more significant liquidity fund as well as an intergenerational fund. Especially in the early years precautionary saving takes up a large share of Iraq's oil rents. Given capital scarcity and inefficient adjustment of public capital, we argue that resource-rich, developing countries like Ghana should aim to use part of their windfall for public investment rather than hedging against commodity price volatility. This gives a boost to their economy and delivers more consumption than a permanent-

[31]For Iraq the windfall is so large that assets per capita do not start to decline until 2050 in the base case if we set $\rho^* = r^*$. The consumption increment in 2012 is now 5.8 times non-resource GDP whilst the size of the windfall in 2012 is 6.5 times non-resource GDP. Given the greater need for precautionary saving as future generations are no longer much richer, consumption in 2012 is reduced by 1.2 times non-resource GDP in the base case (so the total consumption increment is 5.8 - 1.2 = 4.6 times non-resource GDP), thus leaving approximately 70% available for consumption.

[32]Normally, future growth cannot be used as collateral for substantial borrowing, but oil-rich countries such as Iraq have sufficient oil income readily available. Whether we smooth consumption in efficiency units or in per capita terms is ultimately a normative question about how large the social discount rate should be.

Table 2.5: Optimal sizes of intergenerational and liquidity funds (CRP = 3).

Country			Norway	Iraq	Ghana
Intergenerational fund	Final fund size	[% non-resource GDP]	6.8×10^2	1.7×10^4	1.2×10^2
		[$ per citizen]$\cdot\exp(-gt)$	449k	80k	1.7k
	Permanent consumption annuity	[% non-resource GDP]	1.3×10^2	3.3×10^2	2.5
		[$ per citizen per year]$\cdot\exp(-gt)$	8.5k	1.5k	37
Liquidity fund	Additional final fund size	[% non-resource GDP]	3	1.2×10^3	0.2
		[$ per citizen]$\cdot\exp(-gt)$	2.0k	5.5k	2
	Precautionary saving in 2012	[% non-resource GDP]	0.3	7.2×10^1	0.03
		[$ per citizen per year]$\cdot\exp(-gt)$	0.2k	0.3k	0.4
	Additional permanent consumption annuity	[% non-resource GDP]	0.06	22	0.004
		[$ per citizen per year]$\cdot\exp(-gt)$	37	105	0.05

income or bird-in-hand rule. Iraq does suffer much less from capital scarcity, but might have a real problem absorbing its large and growing windfall. In that case Iraq should have a relatively large parking fund. Iraq's main challenge is to deal with a very volatile and growing stream of oil revenues which will continue to make up more than half of GDP for years to come.

Finally, we make four caveats. First, we focused at oil price uncertainty but uncertainty about growth prospects and oil reserves also matter both for the degree of precautionary saving and the rate of oil extraction (e.g. Pindyck (1980, 1981)). Typically, oil extraction is more aggressive either to probe to establish whether there are more, hitherto unknown fields or to have a smaller stock of oil reserves susceptible to oil price fluctuations. Second, Dutch disease effects associated with oil windfalls may matter, especially if the windfall is temporary and not smoothed (e.g. Corden (1984)). The optimal policy should therefore strike a balance between smoothing real exchange rate fluctuations and consumption, investing to invest and mitigating Dutch disease. The latter is tougher if a greater part of consumption and public investment has to be produced at home since adjustment is then more sluggish. Third, we have abstracted from the fact that countries such as Iraq or Ghana may use part of their oil windfalls to expand their extraction capacity. Given that most of this is undertaken by foreign multinationals, capital scarcity should not be an issue if fields are profitable in which case revenue from earlier field should not be used to finance investments in extraction capacity. Fourth, optimal policies need to take account of uncertainty about future asset returns of the sovereign wealth fund within an integrated framework that includes a CAPM model of portfolio investments. To explain the puzzle that oil-rich countries often have large net financial asset positions, yet their trade balances exceed their current accounts, one may need to allow for remittances or the political economy of siphoning off of oil revenue.

Chapter 3

Case study: resource revenues in Alberta

Where Chapter 2 focussed on the resource rents for the economy as a whole, this chapter examines the implications of an uncertain natural resource windfall for government finances in particular. Through a case study aimed at policy makers, based on an equivalent welfare-based intertemporal stochastic optimization model and historical data, this chapter estimates the size of the optimal intergenerational and liquidity funds and the corresponding resource dividend available to the government of the Canadian province Alberta. To first-order of approximation, this dividend should be a constant fraction of total above- and below-ground wealth, complemented by additional precautionary savings at initial times to build up a small liquidity fund to cope with oil price volatility. The ongoing dividend equals approximately 30 per cent of government revenue and requires building assets of approximately 40 per cent of GDP in 2030, 100 per cent of GDP in 2050 and 165 per cent in 2100. Finally, the effect of the 2014 plunge in oil prices on these estimates is examined.

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[JEL E21, E22, D91, Q32]

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3.1 Introduction

The mission of the Alberta Heritage Savings Trust Fund is "to provide prudent stewardship of the savings from Alberta's non-renewable resources by providing the greatest financial returns on those savings for the current and future generations of Albertans." The fund was created in 1976 when 30 per cent of government resource revenue was transferred to the fund. With the economic crises of the early 1980s, this percentage was halved and eventually cut to zero in 1987. Once the Alberta government had eliminated its accumulated debt in 2005 and showed budget surpluses, revenue was again transferred to the fund. Since its inception, $33 billion[2] has been withdrawn from the Alberta Heritage Fund to support spending in health care, education, infrastructure, debt reduction and social programs. The value of this fund stood at $15.1 billion, or 4.7 per cent of Alberta's GDP in March 2014 ($14.9 billion or 4.8 per cent of GDP in March 2013).[3] In addition to this fund, a second, much smaller fund, the Contingency Account, with a value of $4.7 billion or 1.5 per cent of Alberta's GDP in March 2014 ($2.7 billion or 0.9 per cent of GDP in March 2013) is used to smooth revenue arising from volatilities in oil and gas prices.[4] These two funds are examples of what are known in the literature as, respectively, an intergenerational fund and a liquidity fund. We will call the combined total of these two funds simply "the fund."[5]

With fossil fuel extraction rates remaining high for years to come, but the decline in crude oil prices toward the end of 2014 illustrating their inherent uncertainty, the time may be ripe to take a more structural approach to managing Alberta's fund. This chapter argues that it is useful to distinguish between an intergenerational fund to distribute the temporary proceeds from resource wealth over many generations and a liquidity or precautionary savings fund to cushion the adverse impact on government income of a drop in the world price of oil. We use intertemporal stochastic welfare optimization to derive the optimal savings policy. This distinguishes this chapter from Landon and Smith (2015), who use Monte-Carlo techniques to quantitatively compare welfare of several ad-hoc saving rules.

[2] All dollar values ($) reported in this chapter are Canadian dollars, unless indicated otherwise.

[3] We use the book values reported in the annual budget documents by Alberta Finance. Using the slightly higher current fair market value would only marginally affect our calculations and leave our qualitative policy recommendations unaltered.

[4] Given the objective of fiscal stabilization, the contingency account is much more invested in short-term, fixed-income securities than the Heritage Savings Fund.

[5] Both figures come from Alberta's 2014 provincial budget (`http://finance.alberta.ca/publications/budget/budget2014/fiscal-plan-savings-plan.pdf`). The Alberta Government has a number of smaller funds, which include the Medical Research Endowment Fund, the Science and Engineering Endowment Fund and the Scholarship Fund. Their total value is $3.4 billion or 1.1 per cent of Alberta's GDP as of March 2014 ($3.5 billion or 1.1 per cent of Alberta's GDP as of March 2013). We do not include these smaller funds, since they are domestic investment funds. The merit of these funds should be decided on the basis of their social returns. If these returns are satisfactory, Alberta can make use of international capital markets to finance these and not the Heritage Fund.

Our approach is similar to that of Bems and de Carvalho Filho (2011), who examine the effect of precautionary saving on the current account on a number of countries, and van den Bremer and van der Ploeg (2013), who examine Norway, Iraq and Ghana. Specifically, the focus of this chapter is on the implication for government fiscal policy for the Albertan government.

In addition to the recent work by Landon and Smith (2015), van den Bremer and van der Ploeg (2013) and Bems and de Carvalho Filho (2011) discussed above, many authors have studied different aspects of the important question which share of volatile and temporary resource revenues to save, invest and spend and even more have examined its operational policy implications. For example, Barnett and Ossowski (2003) have examined how volatile government resource revenues can lead to the unproductive use of government funds. Based on historical experience, Fasano (2000), Bacon and Tordo (2006) and Kumar et al. (2009) have argued for clear and transparent fiscal rules for payment into and out of a fund. Arrau and Claessens (1992), Engel and Valdés (2000) and Bartsch (2006) among others have used Monte Carlo simulations to assess the performance of stability funds. What sets this chapter apart from this applied policy literature is that it sets out to expose the fundamental economic channel to optimal policy. Ultimately, this relies on the permanent income hypothesis modified for uncertain income to reveal the effect of prudence and precautionary saving (Kimball 1990).

This chapter uses historical data on extraction costs, prices and tax revenues and official projections of extraction rates for Alberta to calculate the size and development of the *optimal* intergenerational and liquidity funds and the corresponding resource dividends, the amount taken annually from the fund and from the resource revenues to be used for general budget purposes. In doing so, we distinguish oil, natural gas and bitumen revenues. How much of the dividend is allocated to public spending, tax cuts or handouts depends on political preferences.[6] The Mintz Commission recommended a target of $100 billion in net financial assets by 2030 and saving a fixed percentage of Alberta's total revenues each year as part of the budget (Alberta Financial Investment and Planning Advisory Commission 2007). Once this target is achieved, the commission foresaw a permanent annual income of $4.5 billion/year to fund public services and/or maintain low taxes in the future. This chapter is aimed at policy makers

Although this chapter focusses on oil and gas price volatility, long-term risk is also based on future, unknown changes in technologies, resource discoveries and transportation investments (e.g. approval of the extended Keystone Pipeline

[6] To strengthen the supply side, one could use the dividend for investment, infrastructure and tax cuts. The Mintz Commission (Alberta Financial Investment and Planning Advisory Commission 2007) dismissed Alaska-style dividend payments as they are lump-sum in nature and have little benefit for the economy. We abstract from the specific allocation of the resource dividend herein, but focus on its optimal size.

System) and uncertainties about future carbon-emission constraints and other policies that impact Alberta's ability to maintain or expand resource production. This chapter's estimates of optimal precautionary saving which only take into account resource price volatility, thus provide a lower bound.

This chapter is laid out as follows. Based on the model in Chapter 2, the model for the optimal management of the intergenerational and liquidity funds used in this chapter is derived and outlined in §3.2 and §3.3, respectively. Our estimates of the optimal sized of these funds for Alberta, based on the data discussed in §3.4, are presented in §3.5. Crucially, §3.6 discusses the sensitivity of our results. Finally, §3.7 concludes.

3.2 How to build an intergenerational fund

Revenue from fossil fuel[7] extraction is temporary, as revenues end when resources are exhausted or too costly to extract, and volatile due to volatile prices. For these reasons, the revenues provide a rationale for an *intergenerational* fund to smooth consumption per capita across generations and a *liquidity* fund to cushion the impact of volatility of the world oil price. We discuss the former first, abstracting from oil price volatility, and the latter in §3.3. We assume a deterministic return on foreign assets r and a fixed marginal cost of extracting one unit of oil. Utility increases at a decreasing rate in the resource dividend D. The government maximizes utilitarian welfare:

$$J(t, F, P, Y) = \max_{D} E_t \left[\int_t^{\infty} U\left(C(\tau)/L(\tau)\right) L(\tau) e^{\rho(\tau - t)} d\tau \right], \tag{3.2.1}$$

where $\rho > 0$ is the social discount rate and L the population size. We explicitly define the resource dividend as the difference between total consumption and non-oil production in the rest of the economy: $D \equiv C - Y$. Non-oil production Y is assumed to be an exogenous process that grows exponentially at a rate of $n + g$, with n denoting population and g productivity growth. Equation (3.2.1) must be solved subject to the budget constraint:

$$\dot{F} = rF + \Omega - D, \quad F(0) = F_0, \tag{3.2.2}$$

where F denotes the fund size and Ω the oil rents. Equations (3.2.1) and (3.2.1) give the Keynes-Ramsey rule for consumption growth:

$$\frac{dC}{dt} = \left[n + \theta(r - \rho)\right] C, \tag{3.2.3}$$

where $\theta > 0$ is the elasticity of intertemporal substitution, having assumed a utility function of the form $U(C) = C^{1-1/\theta} \big/ (1 - 1/\theta)$, and n is the rate of population

[7] Throughout, we refer to 'oil' as a general term to include conventional oil, natural gas and bitumen.

growth. The coefficient of relative intergenerational inequality aversion is $1/\theta$. If we further assume the ratio of consumption and non-oil production is constant in the absence of oil revenues ($\theta(r-\rho) - g = 0$), an assumption discussed further below, we obtain for the resource dividend: $dD/dt = [n + \theta(r-\rho)]\,D$.

By substituting (3.2.3) into the present-value budget constraint and solving, we find that the optimal resource dividend is a constant fraction of total financial and subsoil oil wealth:

$$D(t) = [r - \theta(r-\rho) - n]\,[F(t) + V(t)]\,, \quad V(t) \equiv \int_t^{\infty} e^{-r(\tau - t)}\Omega(\tau)d\tau, \qquad (3.2.4)$$

where oil wealth V is the present value of oil rents. Lower oil extraction costs and larger reserves imply larger oil wealth.

3.2.1 Policy implications

We choose the social discount rate so that the resource dividend and thus the total of financial and oil wealth grow at the same rate as the rest of the economy.[8] Having denoted the per-capita growth rate of non-oil GDP by $g > 0$, non-oil GDP, the resource dividend and total wealth all grow at the rate $g + n$, if we set the social discount rate to $\rho = r - g/\theta < r$. The social discount rate must thus be lower in a growing economy to ensure that more saving occurs and the per-capita resource dividend grows over time. If it is easier to substitute present for future consumption (high θ), this correction term is smaller. From (3.2.4) the propensity to consume out of total wealth is $r - \theta(r-\rho) - n = r - g - n$. Both the resource dividend and total wealth per capita then grow at the rate of productivity growth g. As fractions of GDP they are fully smoothed across different generations.

The *permanent* component of oil revenue is the annuity value of current and future oil revenues, which is the growth-corrected interest on oil wealth $(r - n - g)$ V. The *temporary* component of oil revenue is current minus permanent revenue. If oil revenue is expected to increase (decrease) over time, temporary revenue is negative (positive). The deterministic permanent income hypothesis thus offers the following guidelines for managing resource wealth:

1. The resource dividend that is available to fund the government budget is a constant proportion of total above- and below-ground wealth. It grows at the rate of GDP growth even if oil revenues decline over time, and remains a constant proportion of each generation's non-oil income.

2. The decline in below-ground oil wealth is exactly compensated by an increase in above-ground financial wealth so total wealth remains a constant fraction of total GDP (Hartwick 1977).

[8] Since $\dot{V} = rV - \Omega$ and $\dot{F} = rF + \Omega - D$, with dots denoting time derivatives, we obtain $(\dot{V} + \dot{F})/(F + V) = r - D/(F + V) = \theta(r-\rho) + n$ (from (3.2.4)).

3. The faster the rate of oil depletion and decline in oil revenues, the larger the proportion of revenue that is saved in the intergenerational fund in order that future generations benefit from the current boom in oil revenue. The savings rate out of oil revenues thus varies over time.

3.2.2 Other choices of discount rates

Our pragmatic choice for the social discount rate $\rho = r - g/\theta < r$ has important implications, and alternatives should also be considered. We examine two. First, $\rho = r$ ensures that per-capita consumption is constant and reduces or reverses the rationale for an intergenerational fund in the presence of positive productivity growth. With the prospect of even small productivity growth over an infinite horizon, an incentive arises to borrow heavily to start consuming the permanent value of non-resource GDP now, which goes against the motive to build up savings in the face of declining oil revenues. In the absence of present oil revenue, this borrowing can often not be realized, as it requires borrowing with future growth as collateral. Second, if incumbent politicians try to secure re-election and become impatient, we might have $\rho > r$ so the propensity to consume out of current wealth is higher and the economy saves less and becomes poorer with the passage of time. This effect is less pronounced if politicians have a high willingness to substitute present for future consumption, i.e. a low elasticity of intergenerational inequality aversion (high θ). Although aware of its implications, we proceed under the assumption $\rho = r - g/\theta$, as it allows us to assess the incremental effect of the temporary oil revenues on optimal savings, which would be zero in their absence.

3.3 Oil price volatility and the case for a liquidity fund

To derive the optimal size of the liquidity fund, we extend the model in §3.2 to allow for oil price uncertainty, where the oil price[9] is assumed to follow an autoregressive process with high persistence. The problem is thus to maximize (3.2.1) subject to:

$$\frac{dF}{dt} = rF + \Omega - D, \quad \Omega(t) = \sum_{i=\text{bitumen, crude oil, natural gas}} \left(P_i(t) - \lambda_i\right) O_i(t), \quad (3.3.1)$$

where P_i is the price in \$/barrel of oil equivalent (b.o.e.), λ_i the constant unit extraction cost in \$/b.o.e. and O_i the extraction rate in b.o.e./year. The Keynes-Ramsey rule then becomes:

$$\frac{1}{dt} E_t\left[dD\right] = \left[\theta\left(r - \rho\right) + n\right] D + \frac{1}{2} CRP \left(\frac{D}{D+Y}\right)^2 \sigma_D^2 D, \quad (3.3.2)$$

where CRP denote the coefficient of relative prudence and σ_D the volatility of the dividend (see Appendix A). Prudent policy-making is built on a greater desire to

[9] We adopt three separate correlated price processes for conventional oil, natural gas and bitumen.

avoid negative outcomes than to seek positive outcomes. We have from (3.3.2) with our choice of the discount rate that the dividend as fraction of GDP grows at the rate:

$$\frac{1}{D}\frac{1}{dt}E_t\left[dD\right] - n - g = \frac{1}{2}CRP\left(\frac{D}{D+Y}\right)^2 \sigma_D^2 > 0, \tag{3.3.3}$$

where σ_D is not a constant (see Appendix A).

Hence, the greater the coefficient of relative prudence and the greater the volatility of the dividend, the greater the optimal precautionary buffers that are needed to act as insurance against future drops in oil prices.[10] Furthermore, volatility and the buffers are higher if oil price shocks are less transient, as a greater part of the revenue resulting from shocks is consumed in terms of the resource dividend if these shocks are more permanent thus resulting in larger values for the marginal propensity to consume out of a shock (cf. larger partial derivatives in (A.0.3), see Appendix A for details). If the shocks are permanent (i.e. a random walk), all future oil prices change by the same amount as the initial shock, and the required precautionary buffers are large. If shocks are transient and do not impact future oil prices, very little precautionary saving is required. In other words, with mean reversion in price shocks, the precautionary buffers are smaller.[11] Finally, there is less need for buffers if productivity growth g makes future generations richer and hence better able to deal with future income shocks, as reflected by the ratio of the dividend D and total consumption $C = D+Y$ in (3.3.3).

3.4 Data and assumptions for Alberta

To calculate the optimal intergenerational and liquidity funds and resulting dividends for Alberta, we distinguish between rents from bitumen, conventional oil and natural gas. Although we follow official projections until 2022, we examine two scenarios for the bitumen-extraction paths after that date, where the second scenario is considered in the sensitivity analysis. This section introduces the parameter choice for the base case presented in this chapter[12]. A sensitivity analysis is undertaken in §3.6.

3.4.1 Extraction rates and reserve estimates

For the extraction rates of bitumen, conventional oil, and natural gas, we use official projections available until 2022. In these official projections production of bitumen rises from 0.72 to 1.4 billion barrels per year during the period 2012–2022. Production of conventional oil and natural gas are set to decline from 0.20

[10] Here, precautionary savings are channelled into a fund, but they can also appear as current account surpluses in a small open economy (e.g Bems and de Carvalho Filho (2011).

[11] In contrast, temporary revenue requires more saving in the intergenerational fund.

[12]Further details can be found in Appendix B

and 0.58 to 0.17 and 0.44 barrels of oil equivalent per year,[13] respectively, over the same period. Allowing for some new discoveries, we set initial reserves to 168 billion barrels of bitumen, 4.7 billion barrels of conventional oil and 15.4 billion barrels of oil equivalent of natural gas. Based on these numbers, Figure 3.1 presents two scenarios for the period after 2022. Scenario 1 is the base case scenario. In this scenario, extraction of bitumen continues to increase linearly after 2022 until reaching a value of 2.0 billion barrels per year, remaining constant afterwards until exhaustion.

3.4.2 Government resource rents

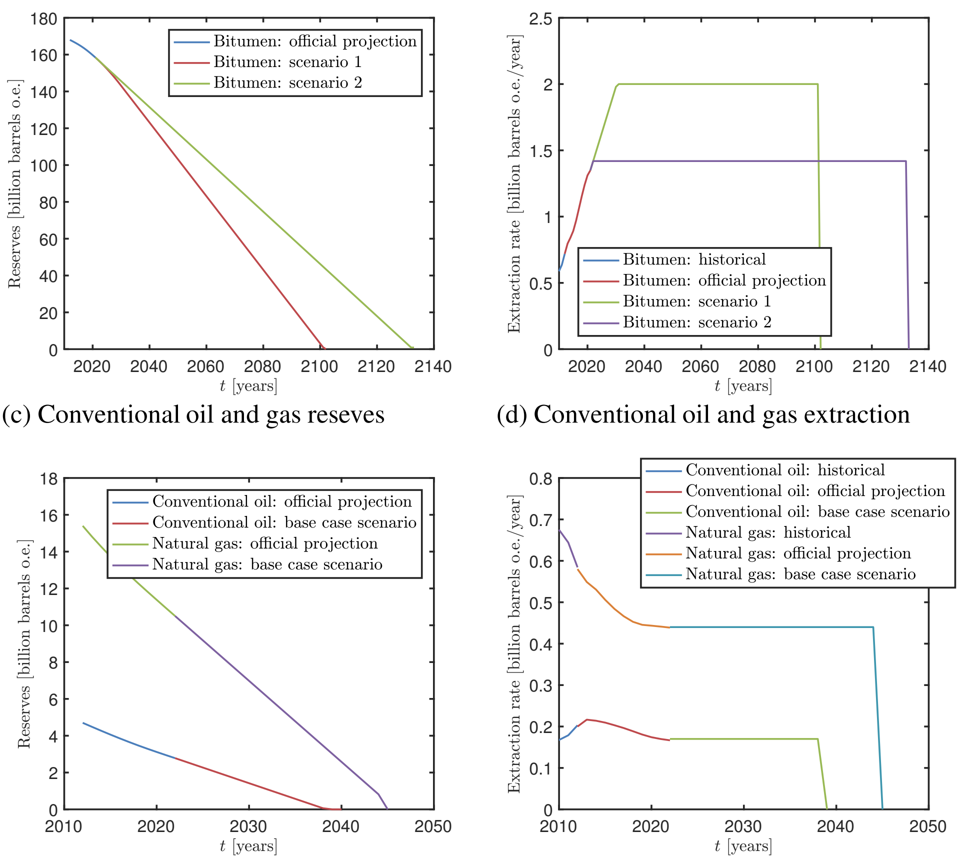
In order to calculate government resource rents, we must first calculate resource rents $\Omega(t) = \sum_i (P_i(t) - \lambda_i)\, O_i(t)$. We use extraction costs of \$15 per barrel of oil equivalent for both conventional oil and natural gas. To reflect the large costs associated with bitumen production, we use an extraction cost of \$32 per barrel. We assume conventional oil is sold at the WTI price and natural gas at the Henry Hub NYMEX natural gas price, but use the much lower (also below Western Canadian Select) average field gate price to estimate the actual price of a barrel of bitumen. For all three resource prices, we adopt AR(1)price processes, reflecting the significant reversion to the mean observed in resource prices. We use the calibration in van den Bremer and van der Ploeg (2013) for the conventional oil and natural gas price with a mean price of \$110 per barrel, a mean reversion of 6 per cent per year, and a volatility of 26 per cent for conventional oil. For natural gas, we take a mean price of \$32 per barrel of oil equivalent, a mean reversion of six per cent per year, and a volatility of 20 per cent. For bitumen, we adopt the same mean reversion and volatility, but a substantially lower mean price of \$80 per barrel.

We assume these prices are perfectly correlated. Initial prices at the start of 2013 are \$96 per barrel natural of oil, \$64 per barrel of bitumen and \$11 per barrel of oil equivalent of natural gas. Extraction of natural gas will initially not be profitable, but becomes profitable when the price reverts back to the mean. If extraction cost exceeds the price of natural gas, gas rents are zero. To reflect the very significant effect the choice of initial (and mean) prices has on our estimates, illustrated, once again, by the drop in prices towards the end of 2014, we consider the effect of such a drop in §3.6.4.

As our focus lies on optimal fiscal policy for the government of Alberta, we assume a constant share of 34 per cent of resource rents accrues to the government

[13] 1,000 bbl of natural gas corresponds to 1.000 bbl of oil equivalent (Norwegian Petroleum Directorate, "Facts: The Norwegian Petroleum Sector" (Oslo: Ministry of Petroleum and Energy, 2011), `http://www.npd.no/en/Publications/Facts/Facts-2011`), which corresponds approximately to equivalent energy content. Under this definition, the per barrel of oil equivalent price of natural gas is significantly lower than the price of oil per barrel, which reflects imperfect substitution and, to a lesser extent, transportation costs.

Figure 3.1: Historical data and projections for extraction rates and reserves.



(a) Bitumen reserves

(b) Bitumen extraction

(c) Conventional oil and gas reseves

(d) Conventional oil and gas extraction

through different taxes and levies, as supported by the data (the 2002–2012 average), thus abstracting from any nonlinearity in the tax regime. Finally, we assume that the share of the non-oil part of government revenue as a share of non-oil GDP is constant at 14 per cent (corresponding to the 2002–2012 average). We report the optimal resource dividend: the increase in government spending that is made possible by the resource revenues.

3.4.3 Return on the fund and general economic trends

The initial size of the fund is \$17.6 billion (both the Contingency Account and the Heritage Savings Trust Fund in March 2013) and is almost 6.0 per cent of total GDP. We set the real return on the fund to r = 6.1 per cent per year (the average annual real return on the Alberta Heritage Savings Trust Fund from 2002 to 2012). We will also present, to verify robustness, our estimates for a lower real return on 4.5 per cent per year in §3.6.2. Trend population growth n is set at 1.3 per cent per year, the long-term projected growth rate for 2014-2041.[14] The trend productivity growth rate g is set at 2.0 per cent per year, so trend growth of non-resource GDP is 3.3 per cent per year. We take an elasticity of intertemporal substitution of θ = 0.5 and thus set the rate of discount to $\rho = r - g/0.5 = 2.1$ per cent per year.

3.5 Optimal intergenerational and liquidity funds for Alberta

Figure 3.1 reports the optimal dividend and size of the fund for extraction scenario 1 for various degrees of prudence. The red line in Figure 3.2a corresponds to the optimal spending/saving mix, expressed as a percentage of government revenue, to build up an intergenerational fund. The red line in Figure 3.2b shows the optimal size of the intergenerational fund, which corresponds to the case without volatility or without prudence. The intergenerational fund grows gradually from 5.7 per cent of GDP in 2013 to 159 per cent in 2100. This sustains an annual dividend between 25 and 31 per cent of government revenue.[15] The purple and blue lines in Figure 3.2 correspond to a moderate (benchmark) and high degree of relative prudence.

3.5.1 Benchmark estimates and the effects of prudence

The optimal initial dividend drops from 28 ($CRP = 0$) to 26 and 21 per cent for degrees of relative prudence of 3 and 10, respectively. The additional initial precautionary saving leads to the build-up of a larger fund with a final fund size in 2110 of 6.5 and 21 percentage points larger for degrees of relative prudence of 3 and 10, respectively. For the benchmark case of $CRP = 3$, the liquidity fund, given by the difference between the $CRP = 0$ and $CRP = 3$ lines, is thus small compared with the intergenerational fund: it grows gradually to a mere 6.5 per cent of GDP in 2100. However, with a much larger relative prudence of 10, the light blue lines indicate that the accumulated liquidity fund is much larger, as reflected by a smaller initial dividend and larger expected resource dividends in the long run.

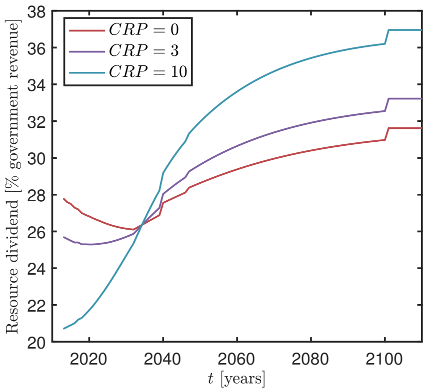
Table 3.1 reports the optimal fund sizes as percentages of GDP and the resource dividends as percentages of government revenue if $CRP = 3$. We also report, in brackets, our estimates for the optimal fund sizes and resource dividends in thousands of 2013 dollars per capita, corrected for productivity growth (the per capita fund sizes grow at the rate of 2.0 per cent per year) and, finally, uncorrected for

[14] Taken from Alberta Finance, Population Projection: Alberta 2014-2041 (2014), `http://finance.alberta.ca/aboutalberta/population-projections/2014-2041-alberta-population-projections.pdf`. In the past, Alberta has seen high rates of population growth with a 10-year average of 2.2 per cent and 20-year average of 2.0 per cent population growth per year (Statistics Canada, 2013).

[15] In fact, the optimal dividend as a share of non-oil GDP is constant in the absence of uncertainty. Variations here merely reflect normalization by total GDP (non-oil + oil GDP), which does not grow at a constant rate unlike non-oil GDP due to changes in the rates of resource extraction.

Figure 3.2: Dividend and fund size with different degrees of prudence (extraction scenario 1).

(a) Bitumen reserves



(b) Bitumen extraction

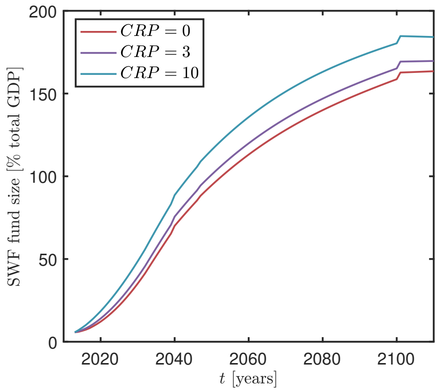


Table 3.1: Estimates of the optimal fund sizes, resource dividends and savings for the Alberta government (*CRP* = 3 and production scenario 1).

	Intergenerational fund (% of GDP)	**Liquidty fund (% of GDP)**	**Total fund (% of GDP)**	**Dividend (% of government revenue)**	**Saving (% of government revenue)**
2013	4.8% ($3.8k pp $3.8k pp)	0.9% ($0.7k pp $0.7k pp)	5.7% ($4.5k pp $4.5k pp)	26% ($2.8k pp $2.8k pp)	3.5%, 2.1% ($0.4k pp $0.4k pp)
2020	12% ($9k pp $11k pp)	1.9% ($1.5k pp $1.8k pp)	14% ($11k pp $13k pp)	25% ($2,.9k pp $3.3k pp)	12%, 1.5% ($1.3k pp $1.5k pp)
2030	35% ($29k pp $41k pp)	4.0% ($3.4k pp $4.7k pp)	39% ($33k pp $46k pp)	26% ($3.0k pp $4.2k pp)	16%, 0.4% ($1.9k pp $2.7k pp)
2050	95% ($72k pp $151k pp)	6.4% ($4.9k pp $10k pp)	101% ($77k pp $161k pp)	30% ($3.2k pp $6.6k pp)	-6.7%, -1.0% (-$0.7k pp -$1.5k pp)
2100	159% ($112k pp $639k pp)	6.5% ($26k pp $148k pp)	165% ($117k pp $665k pp)	33% ($3.2k pp $18k pp)	-28%, -1.6% (-$2.7k pp -$16k pp)

The size of the fund in 2013 is $17.6 billion. The size of resource wealth in 2013 is $1.24 trillion in 2013 or $320k per capita or 400 per cent of GDP. For comparison with the figures in the table, we must multiply this by 0.34, the share of resource rents that accrues to the government, to give $420 billion, $109k per capita, or 137% of GDP. In each cell, the first figure in brackets is in dollars per person. They are corrected for productivity growth and thus grow at 2.0% per year. The second figure in brackets is uncorrected for productivity growth. The figures in the last column report total and precautionary saving as percentage of government revenue; figures in brackets are total saving, growth-corrected and uncorrected.

this growth.

The total fund starts at about \$4.5k per capita in 2013 (5.7 per cent of GDP) and grows to \$33k per capita in 2030 (39 per cent of GDP) and then to \$77k per capita in 2050 (101 per cent of GDP) and \$117k per capita in 2100 (165 per cent of GDP) — all figures in 2013 constant dollars, corrected for growth. This sustains an annual dividend of \$2.8k in 2013 (26 per cent of government revenue) and \$3.2k per capita from 2050 onwards (approximately 30 per cent of public revenue).[16]

This dividend in per capita terms is corrected for productivity growth too, so grows with the rest of the economy at 2.0 per cent per year. This means that the per capita dividend and per capita GDP grow by a factor of 2.1 ($\approx \exp(0.02 \times 37)$) between 2013 and 2050. In real terms, the uncorrected per capita dividend grows from \$2.8k in 2013 to \$6.6k in 2050.

It is instructive to compare our results for Alberta with those for Norway, Iraq and Ghana (van den Bremer and van der Ploeg 2013)). The dividend of \$2.8k per capita is much larger than that for Ghana (U.S.\$37 per capita), larger than that for Iraq (U.S.\$1.5k per capita), but roughly a factor three smaller than that for Norway (U.S.\$8.5k per capita). The optimal final size of the intergenerational and liquidity fund for Alberta reached in 2100 (159 per cent and 6.5 per cent of non-resource GDP, respectively) are rather less than the final fund sizes for Norway (677 per cent and three per cent of non-resource GDP) and very much smaller than those for Iraq (172 and 12 times non-resource GDP), but larger than those for Ghana (115 per cent and 0.2 per cent of GDP). Norway is perhaps the most natural comparison for Alberta. Natural resource revenues last longer in Alberta, and thus there is less need to smooth resource dividends across generations and a smaller intergenerational fund is needed. Comparing to Iraq, it is evident that both windfalls may last for an equally long time, but that they make up a much smaller share of total GDP in the case of Alberta, thus considerably reducing the precautionary motive.

3.5.2 Comparison with the spend-all and bird-in-hand rules

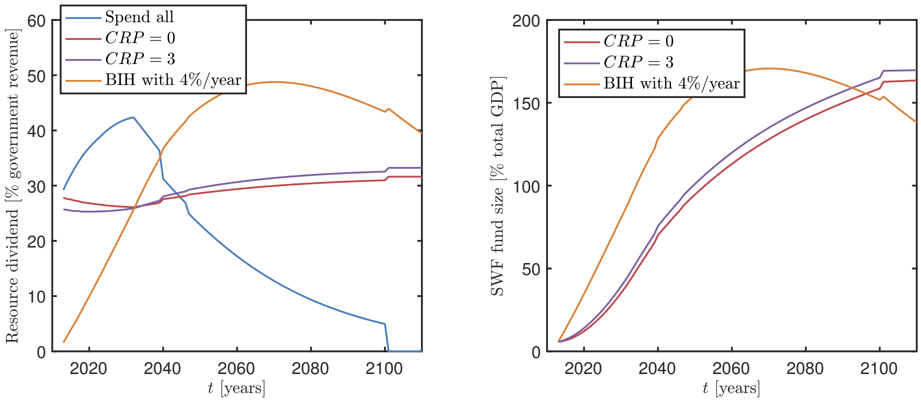
Figure 3.3 compares the benchmark with $CRP = 3$ and the intergenerational fund outcomes corresponding to $CRP = 0$ with a spend-all policy. The blue line denoted by "spend all" shows government resource rents as percentage of total government revenue and thus corresponds to spending all resource rents directly without saving. This spend-all policy is suboptimal for three reasons. First, with excessive spending in the first two decades and a much too rapid decline thereafter not leaving a dividend for future generations, benefits are clearly not smoothed optimally across generations. Second, precautionary buffers are not built up to protect against a future drop in oil prices. Finally, with a significant degree of mean reversion in

[16]Since government revenue as percentage of non-resource GDP is constant and resource rents decline, government revenue as a percentage of total GDP rises slightly.

Figure 3.3: Spend all, permanent-income hypothesis and bird in hand (extraction scenario 1).

(a) Resource dividend

(b) SWF build-up



the oil prices, the resource dividend with a spend-all policy leaves the government budget exposed to extreme volatility.

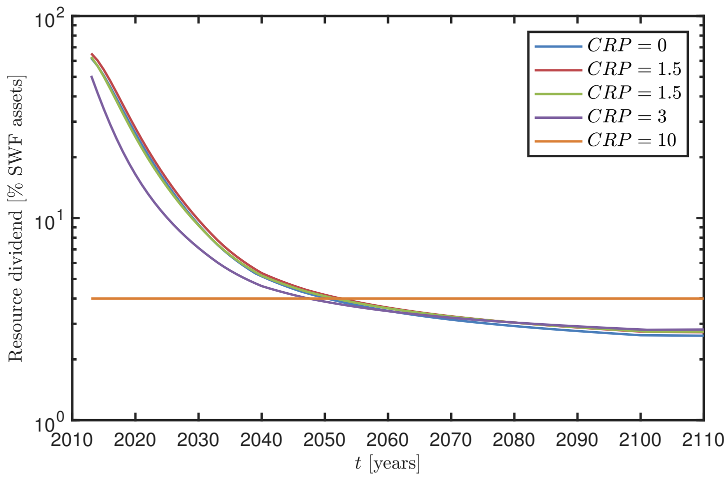
The orange lines in Figures 3.3a and 3.3b illustrate a Norwegian style bird-in-hand (BIH) rule, which does not allow the use of reserves as collateral, puts all resource revenue in the fund and withdraws a fixed 4.0 per cent per year from the fund for general purposes (Bjerkholt 2002, Barnett and Ossowski 2003). We observe that under this rule, wealth is accumulated much more quickly than under the optimal rule, even allowing for the effects of prudence and precautionary savings (i.e. contrasting with the red and purple lines).

Finally, Figure 3.4 shows that, compared with the optimal policy, dividends under the bird-in-hand rule are much too low in the initial periods of the windfall and too high once the windfall has faded away. The optimal policy thus spends a much larger percentage of the fund in the early years and a much lower percentage in later years compared to the bird-in-hand rule. Hence, given substantial amount of below-ground natural resource wealth, it is sub-optimal to set the resource dividend (as Norway does) to a fixed percentage of just above-ground financial wealth.

3.6 Sensitivity analysis

This section discusses the sensitivity of the results presented in the previous section to changes in the production scenario, the real return on assets in the fund, the correlation between oil and gas prices and the initial price level.

Figure 3.4: Resource dividends as percentage of the fund.



3.6.1 Alternative production scenarios

As discussed in §3.2, the timing of the windfall has important implications for optimal savings behaviour. In the benchmark extraction scenario ,1 rents reach a peak of approximately 40 per cent of government revenue in 2030. Such an increase reduces the need for intergenerational saving. In the second scenario, the increase in production of bitumen only continues until reaching a value of 1.4 billion barrels per year (compared to 2.0 billion barrels per year in scenario 1), followed by a similar plateau until exhaustion at a later date, as illustrated in Figures 3.1a and 3.1b. Extraction paths for conventional oil and natural gas, which are set to run out much sooner, are not varied across the scenarios.

The dashed and solid green lines in Figure 3.5a show that the initial optimal spending increment initially drops from 26 per cent in scenario 1 to 23 per cent of government revenue in scenario 2 (with *CRP* = 3). Since, in the alternative scenario 2, the windfall is more spread out over time, a smaller fund has to be built up in the long run. However, more funds have to be accumulated in the short run as production reaches a plateau earlier. The dashed and solid green lines in Figure 3.5b illustrate the effects on the total fund (*CRP* = 3) for the two scenarios.

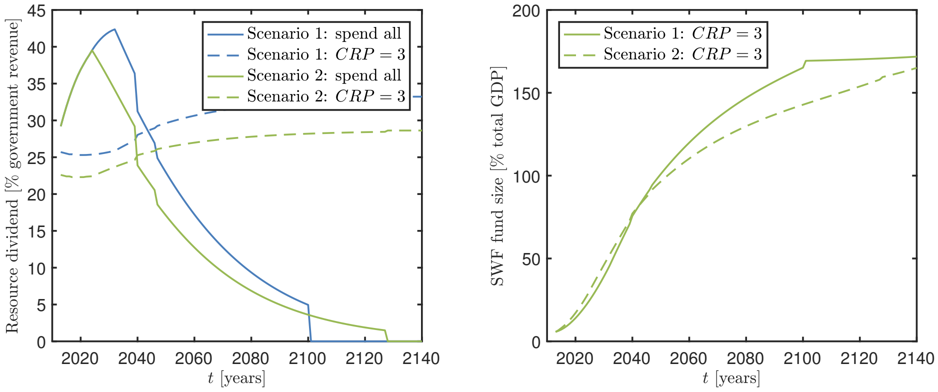
3.6.2 Effects of a lower real return on assets

Figure 3.6 compares the case of a real return on assets of 6.1 per cent per year (base case) based on realized returns by the Alberta Heritage Fund over 2002-2012 to a perhaps more realistic long-term return of 4.5 per cent . Lowering the rate of return, depresses the dividend in the long run, from 33 per cent to 19 per cent of government revenue. It also leads to a greater accumulation of assets and thus to a

Figure 3.5: Optimal spending and build-up of fund for different extraction scenarios (*CRP* = 3).

(a) Resource dividend

(b) SWF build-up



larger fund (Figure 3.6b), as the below-ground wealth that is being converted into above-ground wealth is simply worth more when discounted at a lower rate. The fund size in 2100 is now 215 per cent instead of 165 per cent of GDP.

3.6.3 Correlation between gas and oil prices

Short-term instability of revenue in Alberta can be driven as much by fluctuations in gas prices as by fluctuations in bitumen prices. This is why it is important to stabilize revenue through resource diversification. Empirically, there has been a high degree of negative correlation between oil and gas prices. Although we can allow for such a negative correlation, we find that this does not matter much as rents for natural gas only make up a small part of total resource rents. For example, if the correlation coefficient between gas and oil prices is taken to be -0.5 instead of 1.0, the resource dividend as fraction of public revenue and the fund size as a percentage of GDP are hardly affected, simply reflecting the fact that most revenues are derived from conventional oil and bitumen and not from natural gas.

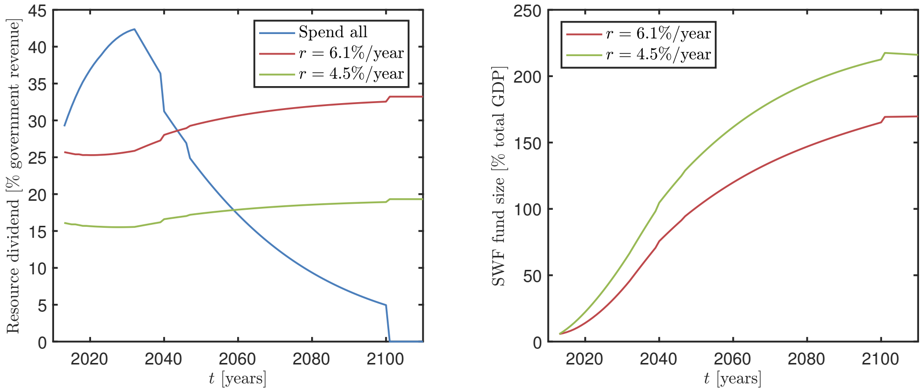
3.6.4 The plunge in oil price

To illustrate the potential effect of a sudden plunge in oil prices, such as the one observed towards the end of 2014, Figure 3.7a shows the initial resource dividend as a function of the initial oil price with Figure 3.7b illustrating the final fund size. Although only the initial conventional oil price is shown on the horizontal axis, we vary either only the initial prices of bitumen, conventional oil and natural gas (continuous lines) or both their mean and initial values (dashed lines) by applying the same scale factor to each. In doing so, we intend to capture the arbitrary nature

Figure 3.6: Effect of a lower real rate of return on fund assets.

(a) Resource dividend

(b) SWF build-up



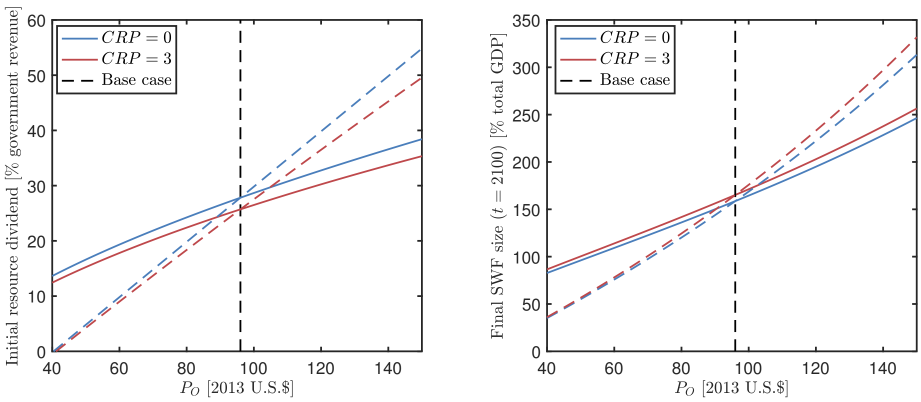
of any initial price assumption of a process with strong random walk characteristics and the lack of robust estimates of the mean price despite evidence of mean reversion and the relative stability of estimates of the rate of mean reversion.

It is evident then from Figure 3.7 that a drop in conventional oil prices to 60 dollars per barrel, reduces the initial resource dividend as percentage of government revenue from 26 to 18 percent of government revenue, with resource revenues dropping from 29 to 7.9 per cent, and cuts the size of the sovereign wealth fund in 2100 from 165 to 114 percent of total GDP. If we also modify the mean prices proportionally, the resource dividend and the size of the fund in 2100 drop even further, to 9.0 and 78 per cent, respectively. Since oil price shocks are very persistent, the size of the resource dividend and the fund that is accumulated varies strongly with the initial oil price that pertains after a truly permanent shock. Compared to our base case, the 2014 plunge implies 30 per cent drop in current resource dividend and final fund size, whereas the drop is a staggering 65 per cent for the dividend and 53 per cent for the final fund size, when the effect is permanent and mean prices also adapt.

Finally, to obtain a sense of the sensitivity to the degree of mean reversion, Figure 3.8 compares the base case with and without mean reversion. As discussed in Appendix A (and Appendix B.5), both a pure random walk and a mean-reverting process for the oil price are difficult to reject on statistical grounds. The continuous lines correspond to the base case discussed in §3.5 with rates of mean reversion of 6.0 per cent for the three price processes, initial prices of \$64, \$96 and \$11 per barrel of oil equivalent for bitumen, conventional oil and natural gas, reverting to mean prices of \$80, \$100 and \$32 per barrel of oil equivalent, respectively. Setting the degree of mean reversion to zero and thus adopting random walk processes for

Figure 3.7: Initial resource dividend and final fund size (t = 2100) as function of initial and mean prices.

(a) Initial resource dividend (b) Final fund size (t = 2100)



In the base case, initial prices are \$64, \$96 and \$11 per barrel of oil equivalent for bitumen, conventional oil and natural gas, reverting to mean prices of \$80, \$100 and \$32 per barrel of oil equivalent, respectively. Although only the conventional oil price is shown on the x-axis for reference, an equivalent scale factor ranging between 0.4 and 1.6 is applied to all three initial prices. For the continuous lines the scale factor is only applied to the initial prices, whereas for the dashed lines the scale factor is applied to both initial and mean prices.

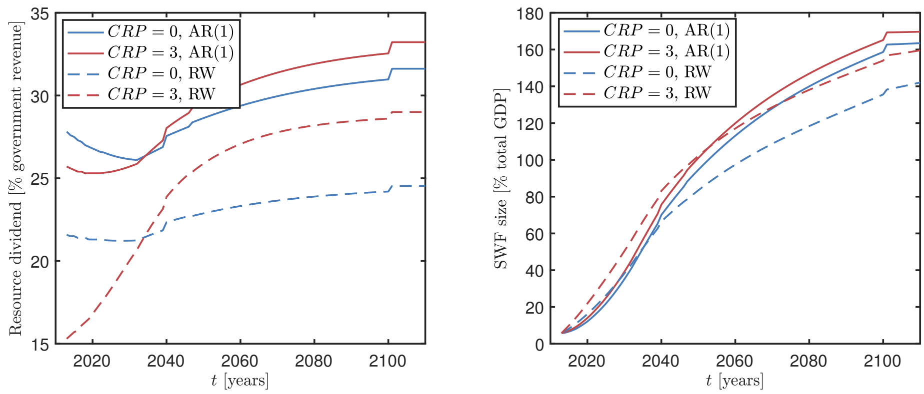
the prices, the dashed lines show the corresponding resource dividends and fund build-up. From the dashed blue lines, it is evident that the absence of a reversion to a higher mean, reduces the final size of the intergenerational fund from 165 per cent of GDP to 136 per cent in 2100. Accordingly, the initial dividend is lower: 22 per cent versus 28 per cent with mean reversion. More importantly, the persistence of shocks now necessitates much greater precautionary savings. At the initial time, the resource dividend drops from 26 to 15 per cent of government revenue and the liquidity fund now constitutes 18 per cent instead of a mere 6.5 per cent with mean reversion at t = 2100.

3.7 Conclusions and policy implications

Following Ossowski (2002), Kneebone (2006) and the Mintz Commission (Alberta Financial Investment and Planning Advisory Commission 2007), our welfare-theoretic analysis examines the optimal savings path of resource revenues in an intergenerational fund to spread the resource wealth across generations and in a liquidity or buffer fund to deal with oil price volatility. We focus our attention on the three main non-renewable resources, bitumen, conventional oil and natural gas, and do not consider renewable resources such as forestry. Crucially, we have chosen the social discount rate such that the optimal resource dividend is a constant

Figure 3.8: Effect of price process (mean reversion vs. random walk).

(a) Resource dividend (b) SWF build-up



fraction of GDP. The per-capita resource dividend thus grows in line with the rest of the economy. Our results suggest that policy in Alberta in can be improved in two ways. Firstly, the amount taken from either the fund or resource revenues for general budget purposes — the resource dividend — should neither be a fixed percentage of financial wealth, as done in Norway, nor should a fixed percentage of annual resource revenues be saved, as recommended by the Mintz Commission[17]. Instead, to first-order of approximation, the resource dividend should be a fixed percentage of the total of above-ground financial and below-ground resource wealth. In the presence of uncertainty, this result is modified slightly, as a small amount of precautionary saving is needed to cope with volatile oil and gas prices. The percentage the resource dividend then makes up out of total wealth is then slightly lower in the short term and higher in the long term reflecting the precautionary motive. Our optimal policies differ from Norway's bird-in-hand rule, which requires that all resource revenues are deposited in the fund and an annual dividend of 4.0 per cent of the fund is withdrawn (e.g. Bjerkholt (2002), Barnett and Ossowski (2003)). As the fund grows, the amount withdrawn increases. Yet, this bird-in-hand rule violates the permanent-income hypothesis and is therefore suboptimal.

Using historical data, we apply our results to the Alberta natural resource windfalls consisting of bitumen, conventional oil and natural gas with 2013 as the start date of our analysis and a corresponding initial oil price of $96 per barrel[18]. Our

[17] More recently, Landon and Smith (2013) advocate a rule that would deposit half of revenues in the fund and set resource dividends at 25 per cent of the fund. Norway deposits all revenues in its fund and withdraws 4 per cent of the fund each year. Although very useful from a policy perspective, such arbitrary rules are suboptimal from a welfare-optimizing perspective across the whole time horizon, must be re-optimized periodically and are never sustainable in the long run.

[18] Our results assume parity between the U.S. dollar, in which oil prices are typically denoted, and

base case estimates suggest that the dividend that can be used to finance government spending or tax cuts is approximately \$2.8k per capita per year in 2013, subsequently growing at 2.0 per cent per year in real terms or, equivalently, at about 30 per cent of total government revenue at all times. Most of the corresponding saving is needed to smooth the dividend as a fraction of GDP. This necessitates a growth in the fund from 5.7 per cent of GDP in 2013 to 39 per cent in 2030, 101 per cent in 2050, and 165 per cent in 2100. In monetary terms, this corresponds to a size of net financial assets of \$46k per capita in 2030 and \$161k per capita in 2050 (both in 2013 dollars, not corrected for growth). This is equivalent to having a target fund in the aggregate of at least \$200 billion by 2030 and \$1 trillion by 2050, compared to the \$17.6 billion that the fund held as of March 2013. The amount that is needed to cushion against oil price volatility - the liquidity fund or Contingency Account - only plays a leading role in the early years, unless policy makers are very prudent.

Although we have abstracted from the stochastic nature of above-ground investments, consideration must be given to the type of investment. The portfolio of assets should be fully diversified, both internationally and across different types of asset groups to minimize risk. The large amount of below-ground resource wealth necessitates that the optimal holdings of risky assets are leveraged up with a factor equal to the ratio of oil wealth to fund wealth, if necessary by going short and taking a negative position in the safe asset (Gintschel and Scherer 2008). The leveraging up of risky assets in the fund's portfolio will be gradually undone as subsoil wealth is depleted (see Chapter 4).

Since Alberta has good access to international capital markets, as illustrated by the very low rates the Canadian government pays on international borrowing (see Appendix B.1), there is no need to spend any part of the fund on public investment projects or to have a separate Alberta Heritage Capital Fund (e.g. Collier et al. (2010), van der Ploeg and Venables (2012)). The decision to invest in domestic capital should be solely based on a cost-benefit analysis, independent of the availability of windfall proceeds, as access to international capital markets guarantees the availability of funds if needed. Moreover, such funds carry the danger of improper calculation of costs and benefits and of political manipulation.

As with all welfare-theoretic analysis stretching across many decades, the figures reported here depend strongly on our assumptions, crucially here on the choice of social discount rate, the return on the fund and the initial oil price. Firstly, our results assume that the resource dividend is indexed to wages and productivity, as is typical for welfare benefits. However, if it is desirable to have a dividend that is

the Canadian dollar, which we use to present our results, based on the situation in 2013. Since 2013 the Canadian dollar has depreciated in value by approximately 20 per cent. Although we have not modelled any such trends nor possible additional volatility due to exchange rates, the depreciation of the Canadian dollar with resource fixed in U.S. dollars would act to increase the value of resource rents and corresponding dividends as expressed in Canadian dollars.

constant in per capita terms, the case for building a large fund is much weakened or even reversed. On the other hand, our estimates for precautionary saving provide a lower bound, the size of the reserves, as future productivity of the non-resource part of the economy, extraction and transportation costs and the long-term cost of carbon emission provide a considerable additional source of uncertainty. Modelling the resource prices as random walk processes, an hypothesis which cannot be rejected statistically, indeed significantly increases optimal precautionary savings, as shown. Our analysis is partial equilibrium in nature, thus takes macroeconomic outcomes and asset returns as exogenous and excludes human capital, or in fact any other types of wealth other than resource wealth, and future pension liabilities. Secondly, if fund managers only achieve a real return of 4.5 per cent per year instead of our benchmark of 6.1 per cent per year, the optimal resource dividend drops from 30 to 20 per cent of government revenue. Yet, the required fund size by the end of this century increases from 165 to 215 per cent of GDP. Finally, if current the plunge in oil prices turns out to be permanent, then our recommendation is to build up a fund of $75 billion by 2030, much more in line with the $100 billion advocated by the Mintz Commission.

Chapter 4

Asset allocation and extraction for resource SWFs

One of the most important developments in international finance and resource economics in the past twenty years is the rapid and widespread emergence of the $6 trillion sovereign wealth fund industry. Oil exporters typically ignore below-ground assets when allocating these funds, and ignore above-ground assets when extracting oil. This chapter presents a unified stylized framework for considering both. Subsoil oil should alter a fund's portfolio through additional leverage and hedging. First-best spending should be a share of total wealth, and any unhedgeable volatility must be managed by precautionary savings. If oil prices are pro-cyclical, oil should be extracted faster than the Hotelling rule to generate a risk premium on oil wealth. This chapter thus generalized the model used in Chapters 2 and 3, which only considered the stochastic nature of (exogenous) resource rents and not that of SWF asset values.

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4.1 Introduction

Since 1994 the number of sovereign wealth funds has nearly quadrupled to 73 (SWF Institute 2013). These funds hold some of the largest portfolios in the world and globally account for over \$6 trillion in assets (ibid.).[2] Two thirds of the sovereign wealth fund industry (by size) has been funded by selling below-ground assets such as oil, natural gas, copper and diamonds ("oil" for short). These funds often comprise a large part of commodity exporters' wealth. Azerbaijan's \$34 billion fund accounts for almost half its GDP, Qatar's \$170 billion fund accounts for almost two thirds of GDP, Saudi Arabia's \$740 billion funds are approximately four-fifths of GDP, Norway's \$840 billion fund is nearly one and a half times GDP, and the United Arab Emirates' \$1 trillion funds are over two and a half times its GDP (SWF Institute 2013, IMF 2013).

The purpose of these funds is to smooth consumption of oil income: across generations because oil reserves are finite, and between periods because oil and asset prices are volatile. While such funds are professionally managed and often allocate their assets using modern portfolio theory, we argue that their investment strategies do not take due account of oil price volatility and subsoil reserves. Similarly, existing theories of optimal oil extraction do not take into account volatile financial markets. These are important issues for resource exporters, since commodity prices are notoriously volatile and below-ground assets can be worth much more than the above-ground fund.

The aim of this chapter is therefore to answer four questions about how below-ground resources should influence above-ground portfolios, and vice-versa. First, how should one allocate above-ground assets given a volatile stock of below-ground assets? Second, how quickly should financial and oil wealth be consumed? Third, how does this change if financial markets are incomplete, so that oil shocks cannot be completely hedged in the portfolio? Finally, how should the optimal extraction rate of below-ground assets be affected by risky above-ground assets?

This chapter will show that policy-makers should adjust their above-ground portfolios to accommodate the volatility and erosion of below-ground oil stocks (hedging and leverage effects respectively); consume a fixed share of total wealth; manage shocks that cannot be hedged with additional precautionary savings; and, if the marginal rent from extracting an additional barrel of oil, namely the oil price minus marginal extraction costs, co-varies positively with average equity market returns, then oil should be extracted faster.

This chapter's analysis combines three large and previously unrelated strands of literature. First, the allocation of financial assets is described by CAPM equa-

[2] All dollars (\$) in this chapter refer to U.S. dollars.

tions modified for subsoil oil wealth. This extends the continuous-time analysis of optimal consumption-saving and portfolio choice (Merton 1990).[3] Second, consumption is described by a stochastic Euler equation,[4] extending the literature on prudence and precautionary savings to the case when both financial assets and oil extraction can be chosen.[5] Third, the optimal rate of oil extraction is described by a stochastic Hotelling rule modified if the proceeds of extraction of below-ground wealth are invested in a risky above-ground financial portfolio.[6] Our intended contribution is to introduce a stylized framework that combines canonical insights from all three of these fields. These insights would be modified by including transaction costs and illiquidity premiums not included in this chapter, which would help to explain why in practice fund managers do not adjust their portfolios too frequently by introducing some mean reversion into the portfolio decisions (Constantinides 1986, Acharya and Pedersen 2005, Garleanu and Pedersen 2013, Jong and Driessen 2015).

This chapter is laid out as follows. First, §4.2 introduces our model for portfolio choice, saving and oil revenues. §4.3 shows how to allow for below-ground oil wealth with a predetermined path for oil production when the oil price is completely spanned by returns in asset markets. §4.4 deals with the case of investment restrictions which prevent the oil price being fully spanned. §4.5 derives the optimal path for oil extraction. §4.6 discusses the implications of our results and compares these with the policies adopted by the Norwegian fund. Finally, §4.7 concludes and qualifies our results.

4.2 The model

Adopting geometric Brownian motion processes for the oil price and asset returns, the problem is to choose the rate of public consumption C and portfolio asset weights w_i, $i = 1,.., n$, to maximize the expected present value of utility with

[3]This builds on classic portfolio theory (Tobin 1958) and mean-variance theory (Markowitz 1952, 1959). If investors have equal information and markets are complete, they hold the market portfolio as used in the CAPM (Sharpe 1964). Our extension to allow for oil income is akin to those dealing with a non-tradable stream of income in the context of university endowments Merton (1993), Brown and Tiu (2012), labour income including endogenous effort (Bodie et al. 1992, Wang et al. 2013), non-tradable and uninsurable income (Svensson and Werner 1993, Koo 1998) and non-financial stores of wealth such as housing Flavin and Yamashita (2002), Sinai and Souleles (2005), Case et al. (2005).

[4]See Leland (1968), Sandmo (1970), Zeldes (1989), Kimball (1990), Carroll and Kimball (2008).

[5]This extends earlier work on precautionary saving in safe assets to cope with oil price volatility (Bems and de Carvalho Filho 2011, van den Bremer and van der Ploeg 2013).

[6]We require marginal extraction costs to be positive and increasing in the amount extracted but, unlike Pindyck (1980, 1981), we do not require them to be convex, which would create extractive prudence. Others treat extraction with stochastic oil prices, growth and capital, but abstract from above-ground financial assets (Gaudet and Khadr 1991, Atewamba and Gaudet 2012). Recent empirical evidence suggests that the Hotelling rule holds at the extensive margin of number of wells drilled, but not at the intensive margin (Anderson et al. 2014, Venables 2014).

discount rate $\rho > 0$:

$$J(F, P_O, t) = \max_{C, w_i} E_t \left[\int_t^\infty U(C(s)) e^{-\rho(s-t)} ds \right], \tag{4.2.1}$$

subject to the budget constraint:

$$dF = \sum_{i=1}^{m} w_i(\alpha_i - r) F dt + (rF + P_O O - C) dt + \sum_{i=1}^{m} w_i F \sigma_i dZ_i, \tag{4.2.2}$$

where the value function $J(F, P_O, t)$ depends on the size of the fund F, the oil price P_O and time t.[7] The rate of oil extracted at time t, $O(t)$, either declines exponentially at the rate κ with zero extraction costs (§4.3 and §4.4)[8] or is chosen optimally with convex costs (§4.5). The fund has m risky assets, $i = 1, ...m$, with drift α_i and volatility σ_i and one safe asset, $i = m + 1$, with return r and volatility $\sigma_{m+1} = 0$. There are thus $n \equiv m + 1$ assets. The fund holds N_i shares of assets, $i = 1, ...n$, each with price P_i, so that $F = \sum_{i=1}^{n} P_i N_i$. The share of each asset in the fund is $w_i \equiv P_i N_i / F$, so that $F = \sum_{i=1}^{n} w_i F$. The stochastic processes for the risky assets are:

$$dP_i = \alpha_i P_i dt + \sigma_i P_i dZ_i, \quad i = 1, .., m, \tag{4.2.3}$$

where dZ_i is a Wiener process with $\mathrm{cov}(dZ_i\ dZ_j) = [\rho_{ij}]$ for i = 1,.., m. The returns of risky assets have covariance matrix $\Sigma = [\sigma_{ij}] = [\rho_{ij}\sigma_i\sigma_j]$. We abstract from mean reversion and stochastic volatility in asset prices, and ignore transaction costs (discussed in §4.6). We thus assume that the coefficients in (4.2.3) are constant. The weight of the safe asset in the fund, $w_n = 1 - \sum_{i=1}^{m} w_i$, is positive or negative if the weight of the risky portfolio is smaller or larger than one, which corresponds to a long position ($w_n > 0$) or short position ($w_n < 0$) in the safe asset. Total holdings of risky assets is called the "portfolio", $(1 - w_n)F = \sum_{i=1}^{m} w_i F$, and its share in the fund is $w \equiv 1 - w_n$.

Preferences exhibit constant relative risk aversion, $U(C) = \left(C^{1-1/\theta} - 1\right)/(1 - 1/\theta)$, $\theta \neq 1$ and $U(C) = \log(C)$, $\theta = 1$, where θ is the coefficient of intertemporal substitution, $1/\theta$ the coefficient of relative risk aversion or the degree of intergenerational inequality aversion, and $1 + 1/\theta$ the coefficient of relative prudence. These are a member of the class of hyperbolic absolute risk aversion preferences and thus permit an analytical solution to the asset allocation problem (Merton 1971). §4.3 also explores Epstein-Zin preferences, which allows one to disentangle risk aversion and intertemporal substitution (Epstein and Zin 1989, Duffie and Epstein 1992).[9]

[7]This abstracts from all other public assets (e.g. future tax revenues) and liabilities (e.g. pensions).

[8]The results can readily be extended for the case of a constant windfall of finite duration.

[9]These have been used in empirical studies (e.g. Attanasio and Weber (1989), Wang et al. (2013)).

The country is a small oil exporter that does not affect the oil price. The world oil price also follows a geometric Brownian motion process:

$$dP_O = \alpha_O P_O dt + \sigma_O P_O dZ_O, \tag{4.2.4}$$

where the drift in the oil price is not too large, $\alpha_O < r$.[10] Again, we abstract from mean reversion, stochastic volatility and transaction costs. Risky assets are driven by a common set of shocks (e.g. to demand, supply, technology or the weather), $du \sim$ i.i.d. $N(0, dt)$. The correlation of each asset depends on how it is affected by these shocks, $dZ = \Lambda du$, where $\Lambda = \left[\lambda_{ij}\right]$ is an invertible $m \times m$ matrix and $dZ = [dZ_1, .., dZ_m]'$ is the vector of Wiener processes driving the returns on risky assets. The Wiener process driving oil returns is expressed as:

$$dZ_O = \lambda_{Oh} du_h + \Lambda_O du = \lambda_{Oh} du_h + M dZ, \tag{4.2.5}$$

where $M = \Lambda_O \Lambda^{-1}$. The vector $\Lambda_O = [\lambda_{O1}, .., \lambda_{Om}]$ determines how the oil price responds to the vector of underlying shocks, du, and $\mathrm{cov}(dZ_O, dZ) = \Sigma M$.[11]

With complete markets, the fund has unrestricted access to all assets and the instantaneous return on oil can be perfectly replicated ("spanned") by a bundle of traded securities. Without loss of generality these securities represent equities and bonds rather than derivatives.[12] The unhedgeable component of oil prices is zero, $\lambda_{Oh} = 0$ (see §4.3 and §4.5). With incomplete markets, there is an unhedgeable component of the oil price with weight $\lambda_{Oh} = \sqrt{1 - \sum_{i=1}^{m-1} \lambda_{Oi}} \neq 0$ and $\Lambda_O = [\lambda_{O1}, .., \lambda_{Om-1}]$, where du_h is a residual oil-specific shock that is uncorrelated with the asset market shocks, du (see Appendix C.1 and §4.4).

4.3 Complete markets and a given path of oil extraction

With complete markets, oil wealth can be treated as tradable by replicating its properties with a synthetic bundle of traded financial assets. Accordingly, an arbitrage argument can be employed to derive the value of the stream of oil revenues (see Appendix C.1 for a derivation):

$$V(P_O, t) = P_O(t)O(t)/\psi, \quad \psi \equiv r + \kappa - \alpha_O + \sum_{i=1}^{m} \beta_i(\alpha_i - r), \tag{4.3.1}$$

where $\beta_i = \frac{\sigma_O}{\sigma_i}[\Lambda_O \Lambda^{-1}]_i$ and $M_i \equiv [\Lambda_O \Lambda^{-1}]_i$. Total wealth, $W = F + V$, then satisfies:

$$dW = \sum_{i=1}^{m} \bar{w}_i W(\alpha_i - r)dt + (rW - C)dt + \sum_{i=1}^{m} \sigma_i \bar{w}_i W dZ_i, \tag{4.3.2}$$

[10]This is a sufficient condition for the present discounted value of a permanent oil windfall to be finite and is consistent with empirical estimates (e.g. van den Bremer and van der Ploeg (2013)).

[11]Oil prices depend in general equilibrium on more fundamental shocks (Bodenstein et al. 2012).

[12]For large oil exporters, liquidity constraints make derivative hedging of oil prices impractical. Therefore, we focus on long/short, equity/bond hedging strategies.

where $\bar{w}_i \equiv (w_i F + \beta_i V)/(F + V),\ i = 1, .., m$.

The replicating bundle linearly combines exposures β_i to many financial assets, which depend on the correlation of each risky asset with the oil price and its uniqueness amongst other financial assets. This bundle matches the variance of oil revenues, and the amount of the safe asset is chosen to match the drift. Oil wealth is current oil revenues divided by the effective discount rate ψ, where ψ is the safe return r plus the rate of decline of oil production κ minus the drift in the oil price α_O plus the adjustment to compensate risk-averse investors for bearing oil price risk.[13] Oil wealth reacts to the current oil price only, as (4.2.4) implies oil price shocks are permanent under our assumptions.

4.3.1 Asset allocation: leverage and hedging demands

If claims to oil can be securitized, the proceeds can be invested in a diversified portfolio, and the problem reduces to that in Merton (1990). In practice, doing so may be difficult due to political and practical constraints.[14] Nevertheless, with the replicating bundle the problem reduces to choosing the net weight of each risky asset, $\bar{w}_i$ for i = 1,.., m, in total above- and below-ground wealth, $W = F + V$. Evidently, the net weight of each risky asset in total wealth is constant:

$$\bar{w}_i = \delta_i \bar{w}, \quad i = 1 \dots m, \quad \delta_i \equiv \frac{1}{v} \sum_{j=1}^{m} v_{ij}(\alpha_j - r), \tag{4.3.3}$$

and the net weight of all risky assets in total wealth is:

$$\bar{w} \equiv \sum_{i=1}^{m} \bar{w}_i = \theta v, \quad v \equiv \sum_{i=1}^{m} \sum_{j=1}^{m} v_{ij}(\alpha_j - r), \tag{4.3.4}$$

where $v_{ij} \equiv \left[\Sigma^{-1}\right]_{ij}$, and the share of safe assets in the total portfolio is $1 - \bar{w}$.

The weight of each risky asset in the above-ground fund is (see Appendix C.2):

$$w_i = \bar{w}_i + \underbrace{\bar{w}_i \frac{V}{F}}_{\text{leverage demand}} + \underbrace{-\beta_i \frac{V}{F}}_{\text{hedging demand}}, \quad \beta_i = \frac{\sigma_O}{\sigma_i} M_i, \quad i = 1, .., m. \tag{4.3.5}$$

Sovereign wealth funds should thus be structured so that net exposure to each asset in total wealth is constant. The optimal portfolio of risky assets (4.3.3) is

[13] The value of an uncertain stream of income follows from discounting at the risk-free rate if the probability space is adjusted to a risk-neutral measure using a theorem due to Girsanov (1960).

[14] Politicians do not like the prospect of having sold oil for an ex-post low price, and risk-averse firms are unwilling to take on all price and production risk

independent of preferences and the level of wealth, but depends as usual on the drift and covariance of asset returns. The optimal part of total wealth allocated to risky assets (4.3.4) is proportional to the overall risk-adjusted return of the portfolio v and the willingness to take risk θ (the inverse of the coefficient of relative risk aversion).[15]

To ensure that net exposure to each financial asset is a constant share of total wealth (4.3.3), one requires offsetting leverage and hedging demands for each risky asset as a share of the above-ground fund (4.3.5). The allocation of the fund approaches its non-oil level, $\bar{w}_i$, as oil is depleted.[16] Leverage demand involves holding more of each risky asset in the above-ground fund. For example, if oil wealth matches the size of the fund and is uncorrelated with assets ($\beta_i = 0, \forall i$, $W = F + V = 2F$), the fund holds twice as much of each risky asset and can do so only by holding less or borrowing more of the risk-free asset. If there is only one risky asset, leverage demand is given by the Sharpe ratio[17] $\theta(\alpha_1 - r)\sigma_1^{-2}(V/F)$, clearly illustrating that, as oil is depleted, leverage demand vanishes by reallocating from risky to safe assets.

Furthermore, hedging demand offsets exposure to oil price risk. If oil is correlated with only one asset, $dZ_O = \rho_{Ok} dZ_k$, hedging demand is the oil-asset beta[18] multiplied by the leverage ratio, $-\rho_{Ok}\sigma_O/\sigma_k(V/F)$. If oil price risk is positively correlated with the financial asset ($\rho_{Ok} > 0$), hedging demand is negative. If the two are negatively correlated ($\rho_{Ok} < 0$), the fund should hold more of the risky asset to hedge oil price risk. Again, as oil is extracted and the exposure to price risk falls, hedging demand vanishes. Equation (4.3.5) generalizes this insight to multiple risky financial assets. If all financial asset returns are independent (Λ is diagonal), oil should be hedged by investing more in assets that are negatively correlated (e.g. assets that use oil as an input such as manufacturing and consumer goods industries) and less in assets that are positively correlated (e.g. oil and gas stocks or substitutes like renewable energy), especially if oil reserves are large. One should then also leverage up all demands for risky assets that prevail in the absence of oil. If financial asset returns are correlated, hedging of oil must consider the covariance of each risky asset. It is then possible that the fund should invest less in assets that are negatively correlated with oil.[19] In practice one can

[15] If there is only one risky asset, reduces to the Sharpe ratio $\bar{w} = \theta(\alpha_1 - r)/\sigma_1^2$, so the portfolio is proportional to the excess return of the risky asset over the safe asset, and the willingness to take risk, and inversely proportional to the variance of the return on the risky asset. With multiple risky assets the overall risk-adjusted return is lower if assets are positively correlated, so there is less scope for fluctuations to offset each other and to hedge oil.

[16] This assumes that withdrawals from the fund are not so rapacious (i.e. ρ is not too high) that fund assets fall quicker than oil is extracted and V/F rises over time.

[17] Mean-variance analysis gives a similar expression (Gintschel and Scherer 2008, Scherer 2009).

[18] The slope coefficient of a regression of demeaned asset returns versus demeaned oil returns.

[19] For example, consider a shock du_G which affects oil and asset A but not others, λ_{OG} , $\lambda_{AG} > 0$ and $\lambda_{iG} = 0$, for all $i \neq A$. The other shocks du_j affect oil and asset A in opposite ways, $\lambda_{Oj} > 0$

implement this with a mix of the "market index" $\bar{w}_i$, and an "oil hedging index" β_i, constructed to replicate movements in the oil price. Over time the mix shifts from the second to the first index as oil is extracted from the ground (see (4.3.5)). Net demand may be negative for both risky assets (short positions) and risk-less assets (leverage), which may not be practical for many SWFs. §4.4 addresses this by considering investment restrictions.

4.3.2 Consumption rules and precautionary saving

Oil wealth also affects precautionary saving and optimal consumption from the fund, as illustrated by the Euler equation governing the expected growth of consumption:

$$\frac{\frac{1}{dt}E_t\left[dC\right]}{C} = \theta(r-\rho) + \frac{1}{2}(1+1/\theta)\sigma_W^2\bar{w}^2, \quad \sigma_W \equiv \sqrt{\sum_{i=1}^{m}\sum_{j=1}^{m}\delta_i\delta_j\sigma_{ij}}. \tag{4.3.6}$$

With complete markets, a closed-form solution for optimal consumption exists (Merton 1990):

$$C = MPC \times W, \quad MPC \equiv r + \theta(\rho - r) + \frac{1}{2}\theta(1-\theta)\left(\frac{\alpha_W - r}{\sigma_W}\right)^2, \quad \alpha_W \equiv \sum_{i=1}^{m}\delta_i\alpha_i, \tag{4.3.7}$$

where the drift and the volatility of total wealth are α_W and σ_W and total wealth also follows a geometric Brownian motion process:

$$dW = \alpha_W^* W dt + \sigma_W \bar{w} W dZ_W, \quad \alpha_W^* \equiv (\alpha_W - r)\bar{w} + r - MPC. \tag{4.3.8}$$

The aggregate volatility of total wealth when portfolio weights are optimised is a weighted average of the volatility of each asset, $dZ_W = \frac{1}{\sigma_W}\sum_{i=1}^{m}\delta_i\sigma_i dZ_i$, and (4.3.8) has solution $W(t) = W(0)\exp\left[\left(\alpha_W^* - \sigma_W^2\bar{w}^2\big/2\right)t + \sigma_W\bar{w}Z_W(t)\right]$.

Aggregate risk is managed by depressing consumption today to build a precautionary buffer of assets, as seen from the upward tilt of the expected consumption path in the final term of (4.3.6). The degree of tilt increases with the coefficient of relative prudence $(1 + 1/\theta)$, the riskiness of the portfolio σ_W^2, and the size of the risky portfolio in total wealth $\bar{w}$. The buffer compensates future periods for bearing additional risk, but does not temporarily support consumption when asset prices are low, as here asset price shocks are random walks and thus persistent.

The optimal spending path can be achieved with a rule that consumes a fixed proportion of below and above-ground wealth, (4.3.7). The proportion is affected

and $\lambda_{Aj} < 0$ for $j = 1, .., m$. It is then possible that oil and asset A are negatively correlated, $\sum_{j=1}^{m}\lambda_{Oj}\lambda_{Aj} < 0$, but the fund should nevertheless invest less in asset A to offset the exposure to shock G. The allocation of all other assets will have to adjust to hedge the effects of the remaining shocks, du_j for $j \neq g$.

by a higher return on the safe asset through the intertemporal substitution effect (negative as future consumption has become cheaper) and the income effect (positive as lifetime wealth has gone up). The former dominates the latter if the elasticity of intertemporal substitution θ exceeds one. From (4.3.7) we see that the marginal propensity to consume, *MPC*, decreases with the return on the safe asset, *r*, and the average excess return on risky assets, $\alpha_W - r$; and increases with relative risk aversion, $1/\theta$, and fund volatility, σ_W. The proportion of total wealth consumed each period, *MPC*, should be less than its expected return $r_e = \bar{w}\alpha_W + (1 - \bar{w})r$, so that both consumption and wealth rise over time.[20] The amount depends on prudence, as $MPC - r_e = -(1/2)(1 + 1/\theta)\,\bar{w}^2\sigma_W^2$, where $1 + 1/\theta$ is the coefficient of relative prudence and we have set $r = \rho$. This precautionary savings builds up a buffer of assets against future risk with absolute risk aversion θ/C falling as consumption rises.

With uncertain oil and asset prices and $r = \rho$, we observe from (4.3.8) how total above- and below-ground wealth evolves over time.[21] It rises due to the premium earned on risky assets, $\alpha_W \geq r$. It falls (rises) if the intertemporal substitution effect is dominated by the income effect in consumption[22] with the extent depending on the risk/return trade-off of total wealth, $-\theta(1 - \theta)\left((\alpha_W - r)/\sigma_W\right)^2 \Big/ 2$.

4.3.3 Intergenerational equity and risk aversion: Epstein-Zin preferences

To capture intergenerational concerns relevant for the long investment horizons of sovereign wealth funds, it is important to separate the coefficient of relative risk aversion, *CRRA*, and the elasticity of intertemporal substitution, *EIS* or the coefficient of relative intergenerational inequality aversion, *IIA* = 1/*EIS* (Epstein and Zin 1989). Restricting attention to one risky and one safe financial asset, we can show that the share of risky assets in total wealth and consumption are $\bar{w} = (\alpha - r)\Big/\left(\sigma^2\ CRRA\right)$ and $C = \left(EIS \times \rho + (1 - EIS)\left[r + (\alpha - r)^2\Big/\left(2\ CRRA\ \sigma^2\right)\right]\right) W$ (see Appendix C.4). These expressions extend (4.3.4) and (4.3.7) by departing from *EIS* = 1/*CRRA* = 1/*IIA*= θ. If *EIS* = 1, the intertemporal substitution and income effects cancel out, so that the propensity to consume is independent of *r*, $C/F = \rho$. If *EIS* > 1 or *IIA* < 1, intertemporal substitution dominates, and the risk-adjusted return in square brackets negatively impacts the propensity to con-

[20] Optimal consumption is a fixed share of total wealth, but also incorporates precautionary saving. Oil is valued at a heavy discount rate but after extraction is replaced with less discounted financial assets, so the value of total wealth and consumption rises over time. Norway takes this to the limit, infinitely discounting future oil revenues and consuming only a fixed share of financial assets (see §4.6).

[21] Without oil or asset price uncertainty and $r = \rho$, any drop in below-ground wealth must be exactly compensated for by an increase in above-ground wealth to fully smooth consumption (Hartwick 1977).

[22] That is, if the elasticity of intertemporal substitution θ is less (greater) than unity

sume. If $EIS < 1$, the income effect dominates and the risk-adjusted return increases the propensity to consume. The Euler equation becomes $(1/dt)\, E_t\, [dC] = \left[EIS \times (r - \rho) + EIS \times CRRA \times CRP \times w^2\sigma^2 \big/ 2\right] C$, where the coefficient of relative prudence equals $CRP = 1 + 1/EIS = 1 + IIA$ for these preferences.

4.4 Investment restrictions and a given path of oil extraction

4.4.1 Additional precautionary saving

Many funds restrict investment in certain asset classes for social and political reasons.[23] This is a form of incomplete markets which prevents the oil price being replicated by a bundle of traded financial securities. To illustrate this, assume that the fund cannot invest in a particular asset, so $\lambda_{Oh} \neq 0$ in (4.2.5) and the oil price is not fully spanned. In that case, there must be additional precautionary saving to cope with residual volatility.[24] With investment restrictions, the Euler equation can be approximated by (see Appendix C.3):

$$\frac{\frac{1}{dt} E_t\, [dC]}{C} = \theta(r - \rho) + \frac{1}{2}(1 + 1/\theta)\left[\sigma_W^2 \bar{w}^2 + \lambda_{Oh}^2 \sigma_O^2 \left(\frac{V}{W}\right)^2\right], \tag{4.4.1}$$

where $\bar{w}$ is given in (4.3.4) and σ_W in (4.3.7). Total wealth evolves according to:

$$\begin{aligned} dW \;\; = \;\; & \left(\textstyle\sum_{i=1}^{m-1} \bar{w}_i W(\alpha_i - r) + \beta_h(\alpha_h - r) + rW - C\right) dt \\ & + \sigma_M \textstyle\sum_{i=1}^{m-1} \bar{w}_i W dZ_i + \sigma_O \lambda_{Oh} V du_O. \end{aligned} \tag{4.4.2}$$

Hence, investment restrictions have both a precautionary and a wealth effect on consumption. The former arises as unspanned risk cannot be hedged optimally, whereas the latter because investment in a specific asset yielding high or low returns is not possible, as such an asset simply does not exist or investment in it is prohibited. Asset weights adjust to find the closest replicating bundle leaving only uncorrelated residual risk (see also Appendix C.1). The precautionary effect describes the additional savings needed because some oil price risk remains unhedged as in (4.4.1). The first term on the right-hand side is the usual slope of optimal consumption. The second term captures precautionary saving and is proportional to the coefficient of relative prudence, $CRP = (1 + 1/\theta)$. The term $\sigma_W^2 \bar{w}^2$ inside the square brackets arises from the precautionary saving needed under complete markets when all oil price volatility is fully diversified. It is proportional to the variance of the portfolio of risky assets and the share of risky assets in the fund squared. The new term $\lambda_{Oh}^2 \sigma_O^2\, (V/W)^2$ arises from the precautionary saving that is

[23] For example, Norway's fund does not invest in tobacco, military or coal assets amongst others.

[24] Earlier work ignored risky financial assets, an extreme case of incomplete markets (van den Bremer and van der Ploeg 2013) (Chapter 2). Here, we have risky assets too, but still allow for incomplete markets.

required if not all oil price volatility can be fully hedged. Less spanning of the oil price (a higher λ_{Oh}) implies that more precautionary saving is required, especially if oil wealth is volatile and comprises a large share of total wealth. Evidently, this effect diminishes as oil reserves are depleted and the ratio of V and W diminishes.

The wealth effect describes the change in the expected return on total wealth from not investing in a particular asset (see (4.4.2)). If an asset cannot be held by the fund (cf. asset h in (4.4.2)), there is still some exposure to it embodied in the oil price. With complete markets this exposure is offset inside the fund, so the net exposure is a constant share of total wealth. With incomplete markets this net exposure cannot be fully offset and will earn a rate of return, changing the expected return on total wealth. Its importance will diminish as oil reserves are depleted.

4.4.2 Stylized illustration of oil-CAPM model

We now illustrate how a sovereign wealth fund is affected by the presence of sub-soil oil, depending on whether or not it has access to hedging assets. We suppose that there is a risk-free asset, r, and two risky assets: *1* uncorrelated with the oil price (the market asset) and *2* perfectly negatively correlated with the oil price (the hedging asset).[25] To ensure the latter asset is used for hedging only, we assume it has a zero excess return. This focuses our attention on the precautionary effect (and sets the wealth effect to zero). Figure 4.1 first gives the declining expected paths of oil revenues and oil wealth and their 95% confidence bounds.

Complete markets

With complete markets there is leverage demand for both risky assets and a hedging demand for asset *2* that is negatively correlated with the oil price, as illustrated by the continuous lines in Figure 4.2. These demands for each risky asset begin large but fall as oil reserves are depleted (cf. $\bar{w}_i V/F$) and exposure to oil prices diminishes.

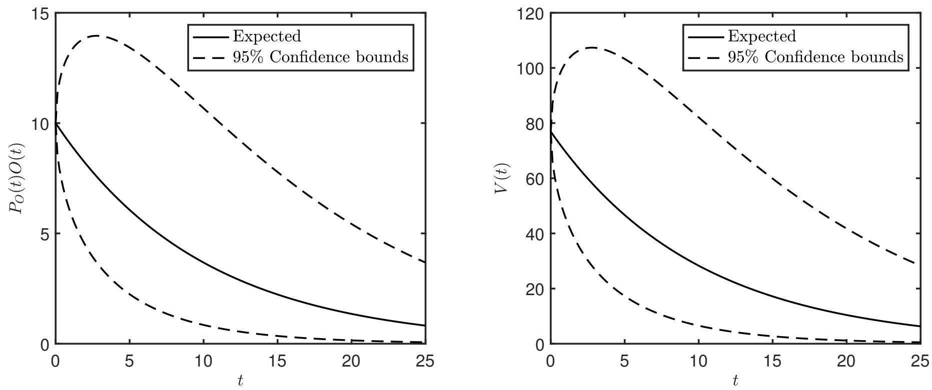
To buy enough of risky asset *2* to fully hedge the oil price, the fund needs to borrow ("short") the risk-free asset. Without oil half of the fund is invested in the risky market asset and the other half in the risk-free asset: $\bar{w}_1 = 0.5$, $\bar{w}_2 = 0$, $\bar{w}_r = 0.5$. Increasing the coefficient of risk aversion will only reduce the demand for the market asset, leaving the hedging demand unchanged, from (4.3.5) and as can be seen from comparing panels (a) and (b).

Figure 4.3 indicates that the consumption path is smoothed in face of declining and volatile oil revenues and grows in line with total above- and below-ground wealth to reflect precautionary saving. As oil wealth is run down (red dotted line

[25] We set $F(0) = 100$, $r = \rho = 0.03$, $\theta = 0.5$ (or $\theta = 0.2$ when indicated), $P_i(0) = 1$, $\sigma_i = 0.02$, $\rho_{ij} = 0$ for $i, j = [1, 2]$; $\alpha_1 = 0.07$, $\alpha_2 = 0.03$, $S(0) = 100$, $O(0) = 10$, $\kappa = 0.1$, $\alpha_O = 0$ and $\sigma_O = 0.25$.

Figure 4.1: Exogenous oil rents and the value of oil.

(a) Oil revenues (b) Value of oil wealth



in panel (b)), the fund is built up (blue dotted line) reflecting the basic insight that total wealth should grow at the same constant rate, if the oil price is completely spanned.

Investment restrictions: incomplete markets

Now consider the situation where the fund is prevented from investing in the risky hedging asset *2* or, equivalently, going short in an asset that correlates positively with the oil price. The dashed lines in Figure 4.2 describe the case with investment restrictions, and indicate that the portfolio weight of the uncorrelated asset *1* is unaffected by restrictions on investing in the hedging asset (see (4.3.3)). The difference arises merely from the change in the drift of the fund F due to the precautionary effect discussed below. By restricting investment in the hedging asset (or, equivalently, preventing short positions in an asset that is positively correlated with the oil price), there is less need to borrow the safe asset (assuming pure hedging assets with zero excess return as in the numerical illustration, thus avoiding wealth effects). Residual volatility will then be managed by additional precautionary savings.

The effect of incomplete markets on consumption is illustrated by Figure 4.4 for the case $CRP = 3$ ($CRRA = 2$). Although not having access to the hedging asset (with zero excess return) does not have a direct effect on the expected evolution of total wealth, it leaves the consumer subject to additional now unhedgeable risk calling for additional precautionary savings. It is clear from panel (a) that initial consumption has to drop in favour of consumption at later times. This effect is larger for larger degrees of prudence, from (4.4.1). Panel (b) shows optimal consumption as a share of total wealth. If oil price risk cannot be hedged due to

Figure 4.2: Portfolio allocation without investment restrictions (solid) and with a ban on investing in the hedging asset 2 (dashed).

(a) Low prudence (*CRP* = 3, *CRRA* = 2) (b) High prudence (*CRP* = 6, *CRRA* = 5)

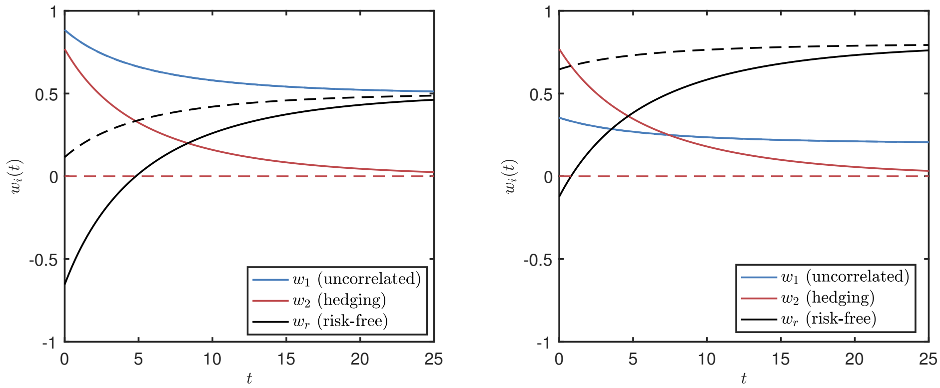


Figure 4.3: Optimal consumption and wealth with complete markets.

(a) Consumption (b) Wealth

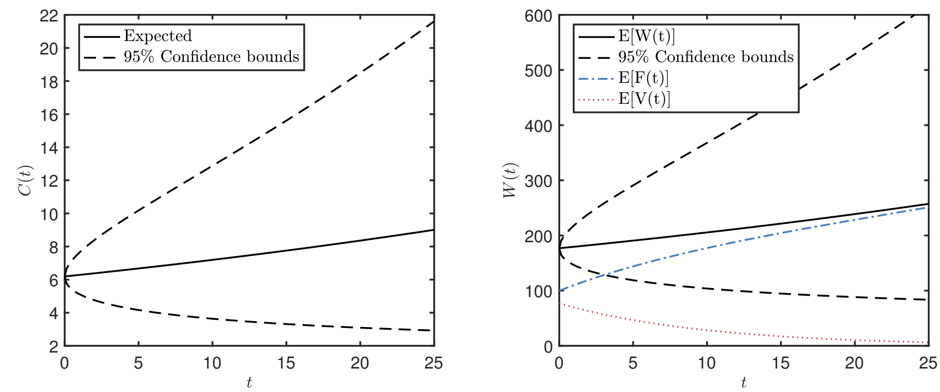
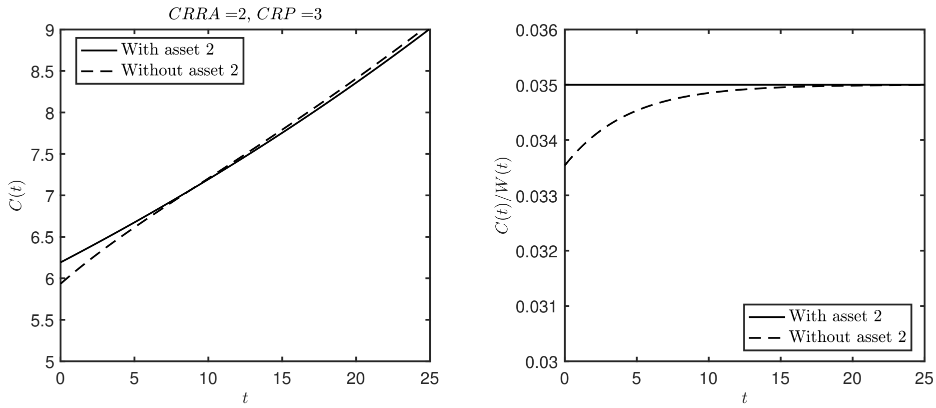


Figure 4.4: Optimal consumption without investment restrictions (solid), with a ban on investing in the hedging asset 2 (dashed).

(a) Consumption (b) Consumption share



incomplete markets or investment prohibitions, the share of consumption in total wealth is no longer constant.

4.5 Portfolio allocation and spending with endogenous oil extraction

The optimal speed of extracting oil may be understood using the Hotelling rule. This states that the expected capital gains from keeping an additional barrel of oil in the ground must equal the return from extracting, selling and earning interest on it (Hotelling 1931). We now extend this rule for volatile oil and financial asset prices.

4.5.1 Optimal rates of oil extraction

Since the data suggest that the oil price is positively correlated with financial assets, we proceed under this assumption.[26] Without loss of generality we assume that the oil price can be perfectly hedged with a single financial asset k, $dZ_O = dZ_k$. The policy maker chooses the consumption rate C, the rate of oil extraction O, and asset weights w_i, i =1, .., m to maximize expected welfare,

$$J(F, P_O, S, t) = \max_{C, w_i, O} E_t \left[\int_s^{\infty} U\left(C(s)\right) e^{-\rho(s-t)} ds \right], \tag{4.5.1}$$

[26] Empirically the extent of this correlation varies over time, as is expected when the source of the oil price shock matters (Kilian 2009). We abstract from this complication here.

subject to the budget constraint:

$$dF = \sum_{i=1}^{m} w_i(\alpha_i - r)Fdt + [rF + \Omega(P_O, O) - C]\,dt + \sum_{i=1}^{m} w_i F\sigma_i dZ_i, \qquad (4.5.2)$$

the geometric Brownian motion processes for asset prices (4.2.3) and oil prices (4.2.4), and the reserve depletion equation:

$$\frac{dS}{dt} = -O(t), \qquad (4.5.3)$$

where oil rents are revenues minus extraction costs, $\Omega(P_O, O) \equiv P_O O - G(O)$, and total extraction costs are increasing in the extraction rate ($G'(O) > 0$) and convex to ensure a solution ($G''(O) > 0$) (cf. Pindyck (1984)) exists. Practically, the assumption of convexity corresponds to costs of extraction for a decision maker at a national level increasing more than proportionally when the rate of extraction is increased.[27] From the depletion equation (4.5.3), cumulative oil extraction cannot exceed initial reserves, $\int_0^{\infty} O(t)dt \leq S_0$. It can be shown (see Appendix C.5) that the optimal path for the expected rate of oil extraction satisfies the modified Hotelling rule:

$$\tfrac{1}{dt}E\,[d\Omega_O] = r\Omega_O + \left(-\frac{\frac{1}{dt}E\,[dJ_F d\Omega_O]}{J_F(F, P_O, S, t)} \right). \qquad (4.5.4)$$

In the particular case of quadratic extraction costs, $G(O) = \gamma O^2/2,\ \gamma > 0$, the stochastic path for oil extraction can be approximated by (for $\alpha_O = 0$):

$$dO \approx \left(-\tfrac{1}{\gamma}\left(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r) \right) P_O + \left(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r) \right) O \right) dt + \tfrac{1}{2} O\sigma_O dZ_O. \qquad (4.5.5)$$

The stochastic Hotelling rule (4.5.4) states that the expected change in marginal oil rents must equal the return on safe assets plus a risk premium. Since we assume that oil and financial asset returns co-move positively, this premium is positive. High oil prices drive high marginal oil rents, which are associated with high fund values, F, and low marginal utility from an extra dollar in the fund($\frac{1}{dt}E\,[dJ_F d\Omega_O] < 0$). The higher return compensates for the risk of holding oil in the ground (equal to $-\frac{1}{dt}E_t[dJ_F d\Omega_O]/(J_F \Omega_O)$). If oil and asset markets are uncorrelated ($\frac{1}{dt}E\,[dJ_F d\Omega_O] = 0$), all oil price risk can be diversified and no risk premium is needed. The more correlated oil and asset markets are, the less oil price shocks can be diversified and the higher the risk premium. Figure 4.5 shows that oil price volatility implies that it is optimal to extract oil initially more quickly. As the rate of extraction drops, extraction costs fall non-linearly boosting the marginal return on oil extraction.

[27] In practice, oil fields evolve stochastically as new fields are discovered and existing fields become economical (e.g. Pindyck (1978)). Extraction costs might be better captured by high upfront investment and small marginal costs. Reserves are also endogenous to exploration effort, but we abstract from these complications here.

Equation (4.5.5) indicates that the optimal rate of oil extraction is positively correlated with the oil price, so that a sudden jump in the oil price requires a jump in the extraction rate to make the most of it. Oil price shocks affect the rate of extraction most when reserves (and in turn O) are highest, since this is when the majority of oil remains exposed to volatile prices. As the date of exhaustion approaches, the rate of oil extraction gets closer to what it would be without volatile oil and asset prices. Note that the size of the fund does not matter for the optimal rate of oil extraction, only the properties of the assets in the background.

Our finding that stochastic oil prices increase the oil extraction rate is consistent with earlier studies, but uses a different mechanism. Earlier work ignored financial assets and relied on "extractive prudence" driven by sufficiently convex marginal extraction costs, $G'''(O) > 0$ (Pindyck 1981).[28] This means it is better to extract oil quickly because, once it is above ground and sold, it is no longer exposed to risk. By restricting our attention to quadratic extraction costs ($G'''(O) = 0$), we deliberately rule out this type of prudence. In our framework oil rents are still exposed to risk above the ground as they must be invested. Hence, oil should be treated as just another part of the total portfolio. The effect of risk on extraction is driven by "extractive risk aversion" ($G''(O)$) rather than by extractive prudence ($G'''(O)$) and so poses less onerous restrictions on extraction costs. Recent literature separates extraction and drilling decisions (e.g. Anderson et al. (2014)). These models also display concavity in either or both choice variable, so risk from above-ground financial markets will still speed up the optimal rate of extraction.[29]

4.5.2 Sovereign wealth funds with endogenous rates of oil extraction

With complete markets and without investment restrictions oil rents can be fully hedged by the fund, regardless of the path of oil extraction. This involves continuously adjusting the asset allocation so that the net exposure to risk remains a constant share of total above- and below-ground wealth. With complete markets oil wealth can be replicated with a bundle comprising the perfectly correlated asset k and the safe asset n, and the value of this bundle evolves according to (see Appendix C.6 for a derivation):

$$dV(t) + \Omega(t)dt = [rV(t) + (\alpha_k - r)\omega_k(O,t)V(t)]\,dt + \omega_k(O,t)V(t)\sigma_k dZ_k(t), \quad (4.5.6)$$

where $\omega_k(O,t) = N_k P_k / V$ is the continuously adjusted share of asset k in the replicating bundle. Total wealth evolves according to:

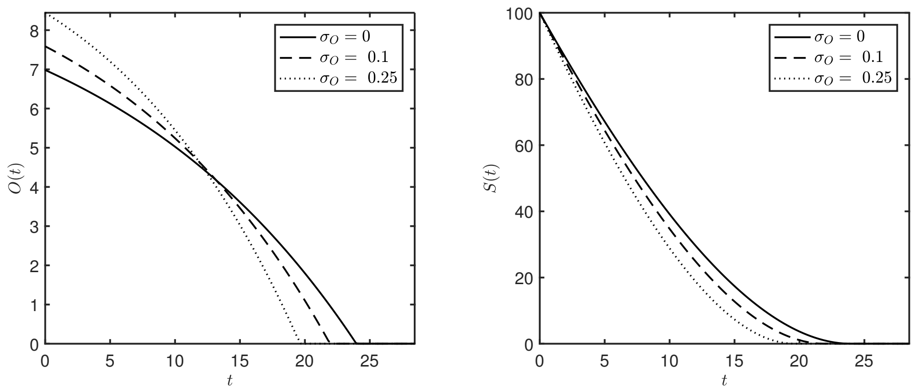
$$dW = \sum_{i=1}^{m}(\alpha_i - r)\bar{w}_i W + (rW - C)dt + \sum_{i=1}^{m}\sigma_i \bar{w}_i W dZ_i, \quad (4.5.7)$$

[28] Aggressive oil extraction also occurs with convex marginal utility arising from market power (van der Ploeg 2010).

[29] Other work estimates oil price volatility from options data and finds that it delays investment in Texas oil wells (Kellogg 2014). However, this relies on a real options argument, whereas we focus on risk aversion and hedging.

Figure 4.5: Endogenous oil extraction.

(a) Optimal rate of oil extraction (b) Subsoil oil reserves



where

$$\bar{w}_i = w_i\,(F(t)/W(t)) \;\; i \neq k, \quad \bar{w}_k = w_k(t)\,(F(t)/W(t)) + \omega_k(O,t)\,(V(t)/W(t))\,. \tag{4.5.8}$$

Oil rents are no longer a geometric Brownian motion as in §4.3, but driven by the drift $\mu_\Omega(P_O, S, t)dt$ and volatility $\sigma_\Omega(P_O, S, t)dZ_O$:

$$d\Omega = \mu_\Omega(P_O, S, t)dt + \sigma_\Omega(P_O, S, t)dZ_O. \tag{4.5.9}$$

The drift and volatility of oil rents are replicated by continuously reallocating the bundle of the perfectly correlated risky asset and the safe asset as P_O and S change. Holdings of asset k in the bundle are adjusted so that the change in oil rents, $\sigma_\Omega(P_O, S, t)dZ_O$, is matched by that in the bundle, $\omega_k(O,t)X(t)\sigma_k dZ_k$. The share of the safe asset is chosen so that the instantaneous drifts also match. As before, the fund is managed to ensure that net exposure to each financial asset is a constant share of total wealth: $\bar{w}_i = \delta_i w_i, \; i = 1,..,m$. Any exposure to asset k embodied in oil, $\omega_k(O,t)$, is offset by the asset's weight in the fund, $w_k(t)$, so that the net weight in total wealth is constant. By rearranging (4.5.8) holdings of each asset in the fund can, as before, be split up into a leveraged and a hedging component for the perfectly correlated asset k:

$$w_i = \bar{w}_i\left(\frac{F+V}{F}\right), \; i \neq k, \quad w_k(t) = \underbrace{\bar{w}_k + \bar{w}_k\left(\frac{V}{F}\right)}_{\text{leveraged demand}} + \underbrace{\left[-\omega_k(O,t)\left(\frac{V}{F}\right)\right]}_{\text{hedging demand}}. \tag{4.5.10}$$

As the asset allocation and consumption problems can be expressed in terms of total wealth (4.5.7), previous results apply for these. Judicious management of the fund allows consumption to be smoothed in line with the permanent income hypothesis and to buffer consumption from oil price volatility by hedging it with traded financial assets.

4.6 Policy implications: Norway's Government Pension Fund Global

The policies of Norway's Government Pension Fund Global (GPFG)[30] closely follow standard CAPM recommendations ignoring oil wealth. Firstly, the GPFG uses the FTSE Global All Cap Index as the equity benchmark (with around 7,400 individual stocks, a close approximation of the market).[31] This is consistent with holding the optimal risky (or market) portfolio if $W = F$ instead of $W = F + V$. Secondly, the Ministry of Finance chooses the equity/bond mix, and in 2007 moved from 40/60% to 60/40%, as it was willing to accept more risk for a higher return. This is consistent with choosing the size of the risky portfolio based on preferences and the overall risk and return of the market, as in (4.3.4) with $W = F$. Thirdly, a fixed share of the fund (4%/year according to Norway's *handlingsregelen*) is consumed each year, as in (4.3.7) with $W = F$.

GPFG's management mandate does not mention oil wealth at all (NBIM 2013), thus leaving Norway exposed to its large and volatile stock of oil wealth: the "elephant in the ground".[32] Norway, and other oil-rich countries with similar funds, would benefit by letting the asset allocation and the consumption rule in the GPFG vary over time.

Norway's asset allocation should vary over time to hedge as much of the volatility of remaining subsoil oil as possible.[33] In the first-best case described in §4.3 this would involve taking large long positions in some industries, and large short positions in others (that may exceed the size of the fund), and reversing these positions as oil is extracted. Such highly leveraged positions expose the country to substantial risk if there are systematic shocks (Das and Uppal 2004). They may also become illiquid, which invalidates the assumption of exogenous prices. Furthermore, the short positions assume that the covariance matrix is stable over time. In practice correlations between oil and each sector vary depending on the type of

[30] At $840 billion the GPFG is the largest single fund in existence, which was established in 1990 to smooth expenditure financed from oil after a period of fiscal volatility in the 1970s and 1980s. Evaluating governance, accountability and transparency, structure and behaviour, the GPFG ranked first on the first two criteria and second overall, behind Alaska's $45 billion permanent fund (Truman 2008), and received the highest rating on the Linaburg-Maduell Transparency Index (SWF Institute 2013). It has been called a "model" for sovereign wealth funds (Chambers et al. 2012, Larsen 2005).

[31] The benchmark is 60% equities, tracking the FTSE Global All Cap Index; up to 5% real estate, tracking the Investment Property Databank's Global Property Benchmark; and up to 40% bonds, of which 70% government and 30% corporate bonds, both tracking Barclays indices.

[32] Norway has proven reserves of nearly 9 billion barrels of oil and 73 trillion cubic feet of natural gas (BP 2014). At 2013 prices these are worth $945 billion and $777 billion, respectively.

[33] Empirical simulations using the correlation of oil prices with financial assets indicate that Norway's exposure to aggregate oil price volatility is halved if oil wealth is hedged in the sovereign wealth fund (Gintschel and Scherer 2008) and that the fund invests less aggressively in risky assets as it ages (Scherer 2009, Balding and Yao 2011). These studies focus on asset allocation but abstract from optimal consumption-saving decisions or oil extraction.

shock hitting the world economy (Kilian 2009). As these correlations can only be estimated using past data and the size of the hedging positions are so large, there is the potential for large basis risk between oil and the hedging portfolio. Finally, as oil is extracted the highly leveraged positions must be reversed which will incur substantial transaction costs for a large fund.[34] Therefore the target index should not be rebalanced too frequently and portfolios should only be adjusted gradually.

A more pragmatic, second-best approach to asset allocation might be to only vary the equity/bonds mix.[35] This would be transparent and easy to explain to investors and the public. It would also not require short positions, have lower transaction costs, and would not rely on a large, time-varying correlation matrix covering all market assets. In this approach, the only risky asset is the overall equity market (e.g. the FTSE Global All Cap Index). If oil is sufficiently positively correlated with this market, the hedging demand to offset oil risk will exceed the leverage demand.[36] In this case, the GPFG should hedge the exposure of subsoil reserves to oil price risk by holding fewer equities and more safe assets while there is oil in the ground. Over time the oil reserves will be depleted and the exposure to equities embodied in subsoil oil will fall. This allows the above ground fund's equity exposure to rise, so that equities make up a greater share of the portfolio as oil is extracted.

The consumption rule should be a constant share of total assets, and thus should fall as a share of the fund as oil is extracted. If oil price risk is perfectly hedged as described in §4.3, this rule should hold exactly. If hedging is imperfect, as would happen by only varying the equity/bond mix, slightly more precautionary savings would be needed. More precautionary saving is also needed if the fund faces a short-sales constraint. Recently, the fund has stopped investing in coal and oil stocks. If the aim is to hedge subsoil oil, it should go further by taking short positions in oil, gas and other stocks that are positively correlated with oil prices. If the aim is to protect the environment, spending should be curtailed to build up a buffer against less diversified risks. In general though, spending as a share of the fund should fall over time as above-ground assets account for an increasing share of total wealth.

These recommendations are relevant for the current debate in Norway. The fund excludes investments in certain assets for social and political reasons, such

[34] See a recent report to the Norwegian Storting (Parliament) (Norwegian Ministry of Finance 2014a).

[35] Gintschel and Scherer (2008) impose short-sale constraints. This does not address the transactions costs that funds face by continuously rebalancing or potentially unstable correlations between assets.

[36] The correlation between the oil price and the overall equity market will also vary over time, though it will be more stable than a covariance matrix covering all 7,400 assets in the FTSE Global All Cap Index. Varying correlations will alter how quickly the equity share in the fund rises. Future work could account for this using regime-switching (cf. Ang and Bekaert (2002).

as tobacco and defence firms, and early 2015 also in assets affected by climate change and other environmental concerns such as coal, oil sands, cement and gold mining. In late 2014 Norway also established a government commission to assess its 4%/year spending rule due to concerns about excessive fiscal stimulus (Norwegian Ministry of Finance 2014b). This follows declining spending as a share of GPFG assets, from nearly 6%/year in 2010 to below 3%/year in 2014, and there have been calls to limit spending to 3%/year in the future (Olsen 2014).

4.7 Conclusions

Commodity exporters have two major types of national assets: natural resources below the ground and a sovereign wealth fund above it. Although some attempts to hedge commodity price volatility have been made, from long-term forward agreements in iron ore until 2010 to the purchase of oil options by Mexico in 2008, there is no evidence of systematic coordination of below- and above-ground assets. We have made the case for coordinating the management of these two types of asset by integrating the theories of portfolio allocation, precautionary saving, and optimal oil extraction under oil and asset price volatility.

This chapter's findings are as follows. Firstly, commodity exporters should change the allocation of their sovereign wealth fund by leveraging all risky assets and hedging subsoil oil risk. These effects are proportional to the ratio of oil and fund wealth, so unwind as resource reserves are depleted. Secondly, consumption should be a constant share of total oil and fund wealth. This stabilizes the mean and variance of spending as total wealth evolves steadily whilst oil reserves are replaced by financial assets, but relies on the degree to which the oil price can be hedged by components of the above-ground portfolio. Thirdly, if oil wealth cannot be adequately hedged, less should be consumed initially in the interests of precautionary savings in the face of the additional unhedgeable risk that remains. Fourthly, the rate of oil extraction should be faster than predicted by the standard Hotelling rule if oil prices are volatile and positively correlated with financial markets. This generates a risk premium on subsoil oil, as convex extraction costs will fall faster than the rate of extraction. The size of the premium will depend on oil's correlation with the market, and disappears to zero if their returns are independent.

This chapter's analysis attempts to offer a first step towards an integrated approach to managing sovereign wealth funds and natural resources under uncertainty. To do this we combine canonical models of asset allocation, precautionary savings and oil extraction. These models, while widely used and theoretically appealing, have received empirical criticism (Griffin 1985, Jones 1990, Fama and French 2004, Anderson et al. 2014). Future work can address this along three dimensions. The first is to analyze the effect of financial assets on natural resources in more detail, allowing for the exploration and discovery of new reserves[37], and extraction decisions at the discrete well level (Kellogg 2014, Anderson et al. 2014, Venables 2014). The second is to extend the analysis to include other non-financial assets such as domestic non-traded capital, human capital and pension liabilities, absorption constraints, general equilibrium effects of spending resource revenues,[38] and the benefits from structural reform to make the economy less vulnerable to commodity price volatility. Finally, there is scope for modelling oil and asset prices in more detail. In practice prices exhibit mean reversion (Wachter 2002), stochastic volatility (Chacko and Viceira 2005, Fouque et al. 2017), large jumps (Ngwira and Gerrard 2007) and time-varying correlations (Bollerslev et al. 1988, Longin and Solnik 1995). Although these extensions allow a better empirical testing of our results, we conjecture that the qualitative nature of our policy insights will be unaffected.

[37] This would extend Pindyck (1978) to a setting with financial assets in order to understand how hedging oil price exposure affects exploration effort.

[38] Gaudet and Khadr (1991) and Atewamba and Gaudet (2012) allow for assets and capital scarcity.

Chapter 5

The risk-adjusted carbon price

A popular model of economy and climate change has logarithmic preferences and damages proportional to the carbon stock in which case the certainty-equivalent carbon price is optimal. This chapter allows for different aversions to risk and intertemporal fluctuations, convex damages, uncertainties in economic growth, atmospheric carbon, climate sensitivity and damages, correlated risks, and distributions that are skewed in the longer run to capture climate feedbacks. This chapter derives a non-certainty-equivalent rule for the carbon price, which incorporates precautionary, risk-insurance and risk-exposure, and climate beta effects to deal with future economic and climatic risks. Quantitative estimates of the risk-adjusted carbon price are obtained after calibration of the model.

The contents of this chapter have been published in working paper form as Van den Bremer, T.S. & van der Ploeg, F. (2018) Pricing carbon under economic and climactic risks: leading-order results from asymptotic analysis. *CEPR Discussion Paper* No. DP12642 and as Van den Bremer, T.S. & Van der Ploeg, F. (2018) The risk-adjusted carbon price. *OxCarre Research Paper* No. 203.[1]

[JEL H21, Q51, Q54]

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5.1 Introduction

Climate policy must take account of the highly uncertain nature of the impact of the atmospheric carbon stock on global mean temperature, of temperature on damages to aggregate output and the highly uncertain nature of future GDP. These uncertainties are in large part due to the long timescales over which today's emissions impact global warming and cause economic damage, but their impact is also affected by skewness of the probability distributions underlying climate uncertainties. To optimally internalize the global warming externality, the price of carbon must be set to the social cost of carbon (SCC),[2] defined as the expected present discounted value of all marginal damages to current and future aggregate production resulting from emitting one ton of CO_2 today.

This chapter's objective is to establish how precautionary and insurance motives affect the optimal risk-adjusted SCC in dynamic stochastic general equilibrium (DSGE). Golosov et al. (2014) provide a pioneering analysis of the optimal SCC in DSGE. Their bold assumptions[3] give a simple rule for the SCC that is proportional to world GDP. However, as they assume logarithmic preferences, economic growth uncertainty does not affect the optimal SCC (Traeger 2017). We generalize their rule for non-unitary elasticities of intertemporal substitution and coefficient of relative risk aversion, generalized convex damages, uncertainty in the rate of economic growth, carbon stock, climate sensitivity and damages, and skewness and mean reversion in the distributions governing climate sensitivity and damage uncertainties. For this purpose, we adapt a DSGE model of endogenous growth with investment adjustment costs due to Pindyck (2013b) to climate change.

Motivated by our search for a tractable rule for the SCC, we use power functions to represent the convex dependence of damages on temperature and the concave dependence of temperature on the carbon stock. Furthermore, by using a power-function transformation of a normal variate displaying a variance that grows in time,[4] we capture the significant right-skew evident in the equilibrium climate sensitivity, but not in the transient climate response, whilst capturing the difference in time scales on which these apply. Although we capture skewness,[5] we deliber-

[2]The first best is sustained in a decentralized market economy if the price of carbon is set to the optimal SCC, either via a carbon tax or a competitive permissions market, provided that the global warming externality is the only market failure. From now on, we will use the optimal (risk-adjusted) carbon price and the SCC interchangeably.

[3]These assumptions are logarithmic preferences, a Cobb–Douglas production function, 100% depreciation of capital in each period, and global warming damages either linear in the atmospheric carbon stock if in the utility function or total factor productivity exponential in the atmospheric carbon stock if in the production function.

[4]Specifically, we use an Ornstein-Uhlenbeck process.

[5]Martin (2013) uses the cumulant generating function to deal with higher moments in the process for the rate of economic growth when analyzing the effects of rare disasters on asset pricing. Pindyck (2013b) also consider skewness and kurtosis of financial markets to model rare disasters.

ately avoid fat tails and thus Weitzman's (2009) 'dismal theorem'.

This chapter obtains the following insights. First, using scaling and analysis of the key non-dimensional quantities of our model, we identify global warming damages as share of world GDP as the only "small" quantity in which to perform a perturbation analysis (cf. Judd 1996, 1998, Judd and Guu 2001).[6] The corresponding optimal price (our Result 1) takes account of an array of uncorrelated and correlated economic, climate and damage risks and can conveniently be evaluated through numerical evaluation of a multi-dimensional integral instead of numerically solving Hamilton-Jacobi-Bellman equations.

Second, by examining only the leading-order effects of various uncertainties, we obtain a leading-order closed-form rule for the optimal SCC, which is proportional to world GDP. This rule is especially simple if the concavity of the temperature-carbon stock relationship is balanced out by the convexity of the damage-temperature relationship so that reduced-form damages are proportional to the atmospheric carbon stock (our Result 2).[7] For a convex dependence of damages on the carbon stock, correction factors can be applied to Result 2 in the form of simple, one-dimensional deterministic integrals (see Result 2′ in Appendix D.1).

From these results, we find that precaution about uncertain economic growth outcomes demands a reduction in the risk-adjusted discount rate and a higher optimal risk-adjusted SCC, especially if risk aversion is high, prudence is high and economic growth is volatile. A positive risk-insurance term acts to increase the risk-adjusted discount rate and increasingly so for more risk aversion and larger economic growth volatility.[8] Provided intergenerational inequality aversion exceeds one, the discount rate is adjusted upwards with positive economic growth and downwards with more volatile growth prospects. The risk-adjusted SCC is adjusted downwards with rising economic affluence and upwards with riskier growth prospects. The correction for climate sensitivity uncertainty depends on the combination of the skewness of its equilibrium probability distribution, the (non-climatic) risk-adjusted discount rate and, crucially, on the time scale on which the skew equilibrium probability distribution is reached.

If future damages and GDP are correlated, there is an effect additional to the

[6] See Bender and Orszag (1999) for an exposition of these techniques for scientists and engineers.

[7] The factor of proportionality between the price and GDP depends on ethical factors (intergenerational inequality aversion, risk aversion and the rate of impatience), economic factors (the rate of growth of the world economy, its volatility and damages to final production from climate change) and geophysical factors (the share of emissions that stays permanently up in the atmosphere and the rate of decay of atmospheric carbon).

[8] If the elasticity of damages with respect to GDP is not unity, the risk insurance premium is multiplied by this elasticity (cf. Dietz et al. 2018). The case of additive damages corresponds to a zero elasticity, in which case there is no risk insurance premium and thus no upward adjustment of the price of carbon due to the insurance effect.

built-in climate beta of one associated with damages being proportional to GDP. If the correlation is positive, this additional "climate-damages beta" is positive. This occurs if a positive climate damage shock (with negative consequences for GDP) is associated with higher economic growth, not through the proportionality of damages to GDP but through the correlation of GDP with the underlying stochastic processes for climate and damages. We show that provided the coefficient of relative risk aversion is greater than one, the risk insurance effect dominates the risk exposure effect and the optimal carbon price is then reduced (cf. Sandsmark and Vennemo 2007, Daniel et al. 2015). Typically, damages are proportional to GDP and thus the climate-damage beta is (close to) one as in Dietz et al. (2018). The latter analyse a "climate beta", which corresponds to an amalgam of the disaggregated climate betas in our model: the built-in climate beta of one associated with the proportionality of damages to GDP and the terms resulting from the correlations of our stochastic processes for climate and damage uncertainties with GDP.

Third, we calibrate our model and decompose the optimal SCC into a deterministic part and parts to deal with uncertain economic growth, carbon stock, climate sensitivity and damages. We thus analyze how ethical discounting, aversion to risk and intertemporal fluctuations, convex damages, skewness of climate sensitivity, and correlated risks affect the components of the optimal risk-adjusted SCC. We find that climate sensitivity and economic growth uncertainties have substantial quantitative impacts on the carbon price, but carbon stock uncertainty has a negligible impact. Climate damage uncertainty is large. Nevertheless, if its distribution is not skewed, it has no effect on the optimal risk-adjusted SCC.

Bretschger and Vinogradova (2018) also analytically examine climate policy in a stochastic environment but focus on the effect of Poisson shocks. Hambel et al. (2017) study numerically the effect of level and growth damages and also separate risk aversion from intertemporal substitution. Jensen and Traeger (2014) show analytically the effect of climate sensitivity on the risk premium in the price of carbon and how this depends on prudence in utility and on the convexity of marginal damages. Although they abstract from a skewed climate sensitivity, they offer an interesting analysis of Bayesian learning of damages. Traeger (2017) puts forward a new IAM with a remarkably detailed climate system and a wide range of objective and epistemological uncertainties, uses cumulant generating functions after transforming to a model that is linear in states with additively separable controls, and shows analytically how the risk-adjusted carbon price depends on the uncertainties. None of these studies analyzes the effects of climate betas.[9]

[9]Earlier work on the effect of uncertainty on the SCC in the IAM developed by Nordhaus (2008) uses Monte Carlo simulations (e.g. Ackerman and Stanton 2012, Dietz and Stern 2015), but this assumes that uncertainty is resolved before the first time period and can give misleading results (Crost and Traeger 2013). Stochastic dynamic programming algorithms do not have this problem, but face limits due to the curse of dimensionality. Recent progress has, however, been impres-

Lemoine (2017) shows that climate sensitivity, damage, consumption, and temperature uncertainties double the optimal SCC, and that the risk insurance effect dominates the risk exposure effect on the optimal SCC if the coefficient of relative risk aversion is greater than one. In the same vain, we find that the sign of the effects of climate betas on the risk-adjusted SCC depends on whether risk aversion exceeds one. However, we give explicit analytical expressions for the SCC. Our approach differs in three other ways. First, we separate the effects of risk aversion and intertemporal substitution, which is crucial in an analysis of the impact of economic and climatic risks on the SCC. Second, we use a fully specified DSGE model rather than an exogenous stochastic process for consumption growth. Third, we allow for skewness and mean reversion in the distributions of climate sensitivity and damages and for more convex damage functions.

Section 5.2 introduces our model. Section 5.3 employs dimensional analysis to identify the key small non-dimensional quantity of our model, applies our perturbation scheme and derives the optimal SCC under uncertainty (Result 1). Section 5.4 makes additional approximations to derive a tractable closed-form rule for the optimal risk-adjusted SCC and explains the rule, including the effects of correlated risks and climate betas (Result 2 and Result 2′ in Appendix D.1). Section 5.5 discusses the calibration of our model. Section 5.6 provides our estimates for the optimal risk-adjusted SCC, including a quantification of the four stochastic determinants and possible climate betas, discusses sensitivity to parameter choice, and assesses the accuracy of our tractable rule for the risk-adjusted SCC. Section 5.7 concludes.

5.2 A DSGE model of global warming and the economy

We use a continuous-time macroeconomic DSGE model with endogenous AK growth model and capital and fossil fuel use as production factors. Fossil fuel use leads to carbon emissions, global warming and damages to aggregate output. We allow for four types of uncertainty: to the growth rate of the economy, the stock of atmospheric carbon, the climate sensitivity, and to global warming damages. We distinguish aversion to risk from aversion to intertemporal or intergenerational differences, so that the coefficient of relative risk aversion, $\eta = CRRA \geq 0$, need not

sive. Traeger (2017) applies stochastic dynamic programming to a 4-state abridged version of DICE. Jensen and Traeger (2014) have Epstein-Zin preferences and study numerically the effect of long-term growth uncertainty. There has also been progress in deriving numerically the optimal SCC when there are tipping risks related to a wide range of catastrophes (e.g. Lemoine and Traeger 2014, 2016, Lontzek et al. 2015, Cai et al. 2016). Lemoine and Rudik (2017) review the numerical literatures on recursive assessment and Monte Carlo evaluation of climate policy under uncertainty, and discuss the importance of learning. Separately, several theoretical contributions have examined the effects of economic growth uncertainty on the optimal discount rate to use for long-term investment projects (e.g. Gollier 2012, Traeger 2017).

equal the inverse of the elasticity of intertemporal substitution *IES*. Alternatively, *CRRA* need not equal the coefficient of relative intergenerational inequality aversion, $IIA = 1/EIS = \gamma \geq 0$. We thus use the continuous-time version of Epstein-Zin (1989) and Kreps-Porteus (1978) recursive preferences following Duffie and Epstein (1992) with the recursive aggregator $f(C, J)$ a function of both consumption C and the value function J, and the rate of pure time preference denoted by $\rho > 0$, so that the representative consumer maximizes

$$J = E_t \left[\int_t^{\infty} f\left(C(s), J(s)\right) ds \right] \quad \text{with} \quad f(C, J) = \frac{1}{1-\gamma} \frac{C^{1-\gamma} - \rho \left((1-\eta) J\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta) J\right)^{\frac{1-\gamma}{1-\eta}-1}}. \tag{5.2.1}$$

The aggregate capital stock is accumulated according to the stochastic equation

$$dK = \Phi(I, K)dt + \sigma_K K dW_1 \quad \text{with} \quad \Phi(I, K) = I - \frac{1}{2}\omega \frac{I^2}{K} - \delta K, \tag{5.2.2}$$

where K denotes the aggregate capital stock, I aggregate investment, $\delta \geq 0$ the depreciation rate of physical capital, and $\omega > 0$ the cost parameter for adjusting investment.[10,11] Adjustment costs are quadratic and homogenous of degree one in capital and investment. Capital is subject to continuous geometric shocks with relative volatility σ_K, and W_1 denotes a Wiener process. Investment is given by $I = Y - C - bF$, where Y is aggregate production, F fossil fuel use, and b the production cost of fossil fuel. Fossil reserves are abundant, and fossil fuel is supplied inelastically at fixed cost. The final goods production function is Cobb-Douglas with constant returns to scale, so $Y = AK^{\alpha}F^{1-\alpha}$ with $0 < \alpha < 1$, where $A \equiv A^*(1 - D)$ is total factor productivity (A^*) net of warming damages DA^*. Damages as share of pre-damages aggregate output D increase in the global mean temperature relative to pre-industrial T. We use the power-function specification

$$D(T, \lambda) = (T/\Delta T)^{1+\theta_T} \lambda^{1+\theta_\lambda} \quad \text{with} \quad \theta_T \geq -1 \quad \text{and} \quad \theta_\lambda \geq -1, \tag{5.2.3}$$

where the stochastic variable λ captures the uncertain nature of damages for a given temperature. The term ΔT is added to ensure that (5.2.3) is independent of units. Henceforth, we define temperature in degrees Celsius and set $\Delta T = 1^{\circ}$C. Damages are a convex function of temperature, which requires $\theta_T \equiv TD_{TT}/D_T > 0$.[12] To allow for potential skewness in the impact of damage shocks, we take a power-function transformation of λ with $\theta_\lambda \neq 0$ and specify a symmetric distribution for λ itself.

[10]In an AK growth model, shocks to the capital stock and shocks to productivity are equivalent. To avoid an extra state, we introduce volatility directly in the capital accumulation equation (cf. Pindyck and Wang 2013).

[11]Sacrificing formality for ease of presentation, we will first introduce the separate evolution equations for the four stochastic variables before introducing the covariance matrix of the vector of these four variables.

[12]We let subscripts denote partial derivatives.

The absolute value of the atmospheric carbon stock is denoted by S. We define the part associated with man-made emissions E as the difference between the current value (S) and the pre-industrial carbon stock, S_{PI}, so that $E \equiv S - S_{PI}$. We let temperature and damages depend on E, not S. Annual carbon emissions from fossil fuel use F are $F\exp(-gt)$, where $\exp(-gt)$ is the emission intensity per unit of fossil fuel used (F, measured in GtC/year), which in accordance with balanced growth we set to decline at the endogenous economic growth rate g. A proportion $0 < \mu < 1$ of fossil fuel emissions ends up in the atmosphere. Atmospheric carbon decays at the rate $\varphi \geq 0$.[13] The dynamics of the carbon stock is given by[14]

$$dE = (\mu F e^{-gt} - \varphi E)dt + \sigma_E dW_2, \tag{5.2.4}$$

where W_2 denotes a second Wiener process, so that the atmospheric carbon stock is described by an arithmetic Brownian motion with absolute volatility $\sigma_E \geq 0$. Note that (5.2.4) ensures that the carbon stock returns to its pre-industrial value in absence of emissions. We have for temperature

$$T(E,\chi) = (E/S_{PI})^{1+\theta_E}\chi^{1+\theta_\chi}\Delta T \quad \text{with} \quad \theta_E \geq -1 \quad \text{and} \quad \theta_\chi \geq -1, \tag{5.2.5}$$

where the stochastic variable χ captures the uncertain nature of temperature for a given carbon stock. As in (5.2.3), $\Delta T = 1^{\circ}$C. The parameter θ_χ allows us to introduce skewness in the impact of stochastic shocks on temperature, and we specify χ itself to have a symmetric distribution. We allow for the effect of lags via the time-varying dynamics of the stochastic process for the random variable χ.[15] Temperature is a concave function of the carbon stock, so that$\theta_E \equiv ET_{EE}/T_E < 0$.[16] The climate sensitivity is defined as the temperature increase from doubling the carbon stock from its pre-industrial level; hence, it is $T_2 \equiv T(E = S_{PI},\chi) = \chi^{1+\theta_\chi}$ and depends on the stochastic climate sensitivity parameter χ. Its normalized skewness $\text{skew}^*[T_2] \equiv \text{skew}[T_2]/(\text{var}[T_2])^{3/2}$ is given to leading-order by $\text{skew}^*[T_2] = 3\theta_\chi(\Sigma_\chi/\mu_\chi)$ (see Appendix D.6), from which it is evident that skewness is driven by what we thus call the skewness parameter θ_χ and the coefficient of variation of $\chi = \Sigma_\chi/\mu_\chi$. Combining equations (5.2.3) and (5.2.5), we get reduced-form damages

$$D(E,\chi,\lambda) = (E/S_{PI})^{1+\theta_{ET}}\chi^{1+\theta_{\chi T}}\lambda^{1+\theta_\lambda} \quad \text{with} \quad \theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi\theta_T. \tag{5.2.6}$$

The parameter $\theta_{ET} \equiv \theta_E + \theta_T + \theta_E\theta_T$ captures the combined effect of the concave relationship between temperature and the carbon stock ($\theta_E < 0$) and the convex relationship between damages and temperature ($\theta_T > 0$). It can thus be positive

[13] One could allow for a permanent reservoir and one (Golosov et al. 2014), two (Gerlagh and Liski 2018) or three (Millar et al. 2016) temporary reservoirs of atmospheric carbon. We show in Section 5.4 that our "1-box" model reproduces historical atmospheric carbon stocks relatively well.

[14] Note that (5.2.4) can theoretically lead to negative carbon stocks, but this is very unlikely.

[15] We thus include the potential effects of temperature lags due to ocean heating, which are important for estimates of the long-run climate sensitivity (e.g. Roe and Bauman 2013) (see Section 5.3).

[16] Temperature is often explained by a logarithmic function of the carbon stock (Arrhenius 1896).

or negative depending on whether the latter effect dominates the former or not. We refer to $\theta_{ET} = 0$ as *proportional* (cf. Golosov et al. 2014) and $\theta_{ET} > 0$ as *convex* (reduced-form) *damages*. The parameter $\theta_{\chi T}$ captures the joint effect of the convex relationship between temperature and climate sensitivity parameter ($\theta_\chi > 0$) and the convex relationship between damages and temperature ($\theta_T > 0$). This parameter is higher if the distribution of climate shocks is more skewed (higher θ_χ). From (5.2.6) total factor productivity and aggregate output decreases in the carbon stock and the shocks to climate sensitivity and damages:

$$Y = A(E,\chi,\lambda)K^\alpha F^{1-\alpha} \quad \text{with} \quad A(E,\chi,\lambda) \equiv A^*\left(1 - (E/S_{PI})^{1+\theta_{ET}}\chi^{1+\theta_{\chi T}}\lambda^{1+\theta_\lambda}\right). \tag{5.2.7}$$

Finally, the uncertainty in the climate sensitivity and damage parameters are driven by two mean-reverting stochastic Ornstein-Uhlenbeck processes with means $\bar{\chi}$ and $\bar{\lambda}$, coefficients of mean reversion ν_χ and ν_λ, and volatilities σ_χ and σ_λ, so[17]

$$d\chi = \nu_\chi(\bar{\chi} - \chi)dt + \sigma_\chi dW_3 \quad \text{and} \quad d\lambda = \nu_\lambda(\bar{\lambda} - \lambda)dt + \sigma_\lambda dW_4, \tag{5.2.8}$$

where W_3 and W_4 are two Wiener processes. Together with $T \propto \chi^{1+\theta_\chi}$ in (5.2.5), the first Ornstein-Uhlenbeck process in (5.2.8) captures the two essential features of the climate sensitivity distribution. First, the transformation $\chi^{1+\theta_\chi}$ of the symmetrically distributed χ allows for positive skewness of the equilibrium climate sensitivity for $\theta_\chi > 0$. Second, uncertainty associated with temperature increases with time, reaching a steady state associated with the equilibrium climate sensitivity and its variance and skewness. We calibrate the properties of the equilibrium climate sensitivity to the steady-state variance ($\Sigma_\chi^2 = \sigma_\chi^2\left(1 - \exp(-2\nu_\chi t)\right)/2\nu_\chi \to \sigma_\chi^2/2\nu_\chi$ as $t \gg 1/\nu_\chi$), so that $1/\nu_\chi$ becomes the e-folding time[18] over which this steady-state is reached. The vector of all four states can be described by one multi-variate Ornstein-Uhlenbeck process:

$$d\mathbf{x} = \boldsymbol{\alpha} - \boldsymbol{\nu} \circ (\mathbf{x} - \boldsymbol{\mu})\, dt + \mathbf{S}d\mathbf{W}_t, \tag{5.2.9}$$

where $d\mathbf{x} \equiv (dk, dE, d\chi, d\lambda)^T$, $k \equiv \log(K/K_0)$ and $\circ$ denotes the elementwise product of two vectors. The growth rates of this process are

$$\boldsymbol{\alpha} \equiv \left(\frac{1}{dt}\frac{E_t\,[dK]}{K} - \frac{1}{2}\sigma_K^2, \mu F e^{-gt}, 0, 0\right)^T. \tag{5.2.10}$$

[17]For independent stochastic processes, (2.8) has solution $\chi(t) = \chi_0 e^{-\nu_\chi t} + \bar{\chi}(1 - e^{-\nu_\chi t}) + \sigma_\chi \int_0^t e^{-\nu_\chi(t-s)}dW_3(s)$, and similarly for the stochastic process for λ. The random variables $\chi(t)$ and $\lambda(t)$ are normally distributed with time-varying moments: $\chi(t) \sim \mathrm{N}(\mu_\chi, \Sigma_\chi^2)$ and $\lambda(t) \sim \mathrm{N}(\mu_\lambda, \Sigma_\lambda^2)$. Mean and variance of $\chi(t)$ are $\mu_\chi = \chi_0 e^{-\nu_\chi t} + \bar{\chi}(1 - e^{-\nu_\chi t})$ and $\Sigma_\chi^2 = \sigma_\chi^2\left(1 - \exp(-2\nu_\chi t)\right)/2\nu_\chi$ with stationary limits $\mu_\chi \to \bar{\chi}$ and $\Sigma_\chi^2 \to \sigma_\chi^2/2\nu_\chi$.

[18]This is the time it takes for an exponentially growing quantity to increase by a factor e = 2.71828.

The vector of mean reversion rates and the vector of means of this process are

$$\boldsymbol{\nu} \equiv (0, \varphi, \nu_\chi, \nu_\lambda)^T \quad \text{and} \quad \boldsymbol{\mu} \equiv (0, 0, \bar{\chi}, \bar{\lambda})^T. \tag{5.2.11}$$

The covariance matrix $\mathbf{SS}^T$ of the components of this multivariate process is

$$\frac{1}{dt}E_t\left[d\mathbf{x}d\mathbf{x}^T\right] = \mathbf{SS}^T = \begin{pmatrix} \sigma_K^2 & \rho_{KE}\sigma_K\sigma_E & \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{K\lambda}\sigma_K\sigma_\lambda \\ \rho_{KE}\sigma_K\sigma_E & \sigma_E^2 & \rho_{E\chi}\sigma_E\sigma_\chi & \rho_{E\lambda}\sigma_E\sigma_\lambda \\ \rho_{K\chi}\sigma_K\sigma_\chi & \rho_{E\chi}\sigma_E\sigma_\chi & \sigma_\chi^2 & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda \\ \rho_{K\lambda}\sigma_K\sigma_\lambda & \rho_{E\lambda}\sigma_E\sigma_\lambda & \rho_{\chi\lambda}\sigma_\chi\sigma_\lambda & \sigma_\lambda^2 \end{pmatrix}, \tag{5.2.12}$$

where ρ_{ij}, $i \neq j$, $i, j = K, E, \chi, \lambda$ denote the partial correlation coefficients.

5.3 Asymptotic solutions for the optimal risk-adjusted price of carbon

The optimal solution under uncertainty satisfies the Hamilton-Jacobi-Bellman equation corresponding to the recursive utility specification (5.2.1) which is

$$\max_{C,F}\left[f(C,J) + \frac{1}{dt}E_t\left[dJ(t,K,E,\chi,\lambda)\right]\right] = 0, \tag{5.3.1}$$

where $(1/dt)\,E_t\,[dJ]$ is Ito's differential operator applied to J, which depends on time and the four states. Using $I(C,F,K,E,\chi,\lambda) = A(E,\chi,\lambda)K^\alpha F^{1-\alpha} - C - bF$ and applying Ito's calculus to (5.3.2) gives

$$\begin{aligned} &\max_{C,F}\left[f(C,J) + J_K\Phi\left(I(C,F,K,E,\chi,\lambda),K\right) + J_E(\mu F e^{-gt} - \varphi E)\right] + J_t \\ &+ J_\chi\nu_\chi(\overline{\chi} - \chi) + J_\lambda\nu_\lambda(\overline{\lambda} - \lambda) + \frac{1}{2}J_{KK}K^2\sigma_K^2 + \frac{1}{2}J_{EE}\sigma_E^2 + \frac{1}{2}J_{\chi\chi}\sigma_\chi^2 + \frac{1}{2}J_{\lambda\lambda}\sigma_\lambda^2 \\ &+ J_{KE}K\rho_{KE}\sigma_K\sigma_E + J_{K\chi}K\rho_{K\chi}\sigma_K\sigma_\chi + J_{K\lambda}K\rho_{K\lambda}\sigma_K\sigma_\lambda + J_{E\chi}\rho_{E\chi}\sigma_E\sigma_\chi \\ &+ J_{E\lambda}\rho_{E\lambda}\sigma_E\sigma_\lambda + J_{\chi\lambda}\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda = 0. \end{aligned} \tag{5.3.2}$$

By differentiating (5.3.2) with respect to the forward-looking variables C and F, we obtain the optimality conditions

$$f_C = C^{-\gamma}\left((1-\eta)J\right)^{\frac{\gamma-\eta}{\eta-1}} = J_K\Phi_I(I,K) \tag{5.3.3}$$

(using (5.2.1)) and $(1-\alpha)Y/F = b + Pe^{-gt}$, where the optimal risk-adjusted SCC is defined by $P \equiv -\mu J_E/J_K\Phi_I(I,K) > 0$. Although solving our problem as a command optimum, the solution corresponds to the outcome in a decentralized market economy provided carbon emissions are priced at an amount equal to the SCC and that there are no other externalities or market failures. Henceforth, we will therefore use the price of carbon and the SCC interchangeably, and we denote these by P.

5.3.1 Transforming to non-dimensional form and scaling

Closed-form analytical solutions to the stochastic dynamic optimal control problem (5.3.1)-(5.3.2) are not available. Solving this numerically by approximating the value function and its derivatives in 5-dimensional (time and the four states) space is challenging due to the curse of dimensionality and will not give the insight into the mechanisms that determine the stochastic markup of the optimal price of carbon we seek. Instead, we first examine the system for small parameter(s) by transforming to non-dimensional form, then pursue an asymptotic expansion in the thus identified small parameter(s) and only consider leading-order. Out of all the non-dimensional groups (see Appendix D.2), we identify one group to be small:

$$\epsilon \equiv \overline{\lambda}^{1+\theta_\lambda} \overline{\chi}^{1+\theta_{\chi T}} \left(E_0/S_{PI}\right)^{1+\theta_{ET}} , \tag{5.3.4}$$

where E_0 denotes the difference between the absolute value of the atmospheric carbon stock at $t = 0$ (S_0) and the pre-industrial carbon stock S_{PI}. The non-dimensional group ϵ becomes the small parameter of our asymptotic expansion, whereas the effects of the other non-dimensional groups are initially analyzed without approximation. The quantity ϵ represents the share of climate damages in total GDP (when climate damage and sensitivity parameters are at their equilibrium values and the atmospheric carbon stock at its initial value). It is typically only a few percentage points and stays well below 10% even at high temperatures (see Section 5.4.3). In the perturbation solutions below, we consider terms up to first order in ϵ. The resulting error scales with $\epsilon^2 \cong 0.01$, which is small even for the large value of $\epsilon \cong 0.1$, and we deem this to be more than sufficient for accurately estimating the optimal risk-adjusted SCC.

5.3.2 Perturbation expansion

To solve our problem, we perform a perturbation expansion[19] in the small parameter ϵ. At each order n, the problem is linear in the value function $J^{(n)}$ but remains fully nonlinear in the states, thus retaining risk-aversion and prudence properties without approximation. Mathematically, at each order n, the problem is of the form $L[\epsilon^n J^{(n)}] = \Gamma$ where L is a linear differential operator in the states and the nonlinear forcing Γ is formed from products or derivatives of lower-order solutions (in n), so that the order of the forcing thus obtained (from products or derivatives) is also $O(\epsilon^n)$. We choose the following truncated series solution and restrict our attention

[19] We emphasize we do not perform a Taylor-series expansion in the state variables around their steady states, as this requires an excessive number of terms due to the large number of states and derivatives needed to capture risk aversion and prudence.

to zeroth- and first-order terms in ϵ only, as denoted by the superscripts

$$\begin{aligned} J(K,E,\chi,\lambda,t) &= J^{(0)}\left(K,\epsilon D(E,\chi,\lambda)\right) + \epsilon J^{(1)}(K,E,\chi,\lambda,t,\epsilon D(E,\chi,\lambda)) + O(\epsilon^2), \\ F(K,E,\chi,\lambda,t) &= F^{(0)}\left(K,\epsilon D(E,\chi,\lambda)\right) + \epsilon F^{(1)}(K,E,\chi,\lambda,t,\epsilon D(E,\chi,\lambda)) + O(\epsilon^2), \\ C(K,E,\chi,\lambda,t) &= C^{(0)}\left(K,\epsilon D(E,\chi,\lambda)\right) + \epsilon C^{(1)}(K,E,\chi,\lambda,t,\epsilon D(E,\chi,\lambda)) + O(\epsilon^2). \end{aligned} \tag{5.3.5}$$

The small parameter ϵ appears both as a small parameter of the series solution and as the multiple-scales parameter in front of the dependence on damages. We let total factor productivity be a slowly-varying power-law function of the climate-related variables E, χ and λ: higher derivatives required to model rapid variation are thus automatically ignored at leading order. The zeroth-order value function inherits the properties of the production function (5.2.7). We then find a consistent leading-order estimate of the SCC from the zeroth and first-order value function:

$$P = -\mu\left(J_E^{(0)} + \epsilon J_E^{(1)}\right)/\phi'(i^{(0)})J_K^{(0)} \ . \tag{5.3.6}$$

In the limit $\epsilon \to 0$, climate has no effect, we retain only the zeroth-order solution, and our model reduces to an AK model for which a closed-form solution is available (cf. Pindyck and Wang 2013). Our derivation of the zeroth-order solution is given in Appendix D.3 with the solution for $J^{(0)}$ given by (D.3.4) (written in terms of the non-dimensional variables introduced in Appendix D.2). The only difference with Pindyck and Wang (2013) is that ours depends slowly on the climate variables, as determined by the implicit equation for optimal investment (D.3.7) and the dependence of the marginal productivity of capital therein on climate damages. We proceed to derive the first-order solution in Appendix D.4 with the solution for $J^{(1)}$ given by (D.4.27).[20] The first-order value function captures changes to the economy resulting not from climate-induced changes to the marginal productivity of capital (as captured by $J^{(0)}$'s slow dependence on the climate-related states), but from direct damages to the economy arising from the three climate-related states.

5.3.3 Perturbation solutions

Combining the zeroth- and first-order solutions, we get the following result.

Result 1: The optimal risk-adjusted SCC is (cf. (D.4.32)):

$$P = \frac{\mu\,\Theta\,(E,\chi,\lambda)\,Y|_{P=0}}{r^*}\left(1 - \frac{\Omega(K,E,\chi,\lambda,t)}{E^{\theta_{ET}}\chi^{1+\theta_{ET}}\lambda^{1+\theta_\lambda}K^{1-\eta}}\right) + O(\epsilon^2), \tag{5.3.7}$$

where $\Theta \equiv D_E\,(E,\chi,\lambda)\,/\,(1-D\,(E,\chi,\lambda))$ denotes what we call the flow damage coefficient, the zeroth-order return on capital corrected for growth is

$$r^* \equiv r^{(0)} - g^{(0)} = \rho + (\gamma-1)(g^{(0)} - \eta\sigma_K^2/2), \tag{5.3.8}$$

[20]We only show the solution for $J_E^{(1)}$ as this is needed to evaluate the SCC.

and $r^{(0)}$ and $g^{(0)}$ are the leading-order expected rates of return on investment and economic growth. Furthermore, we have

$$\Omega = E_t \left[\int_t^{\infty} \Gamma(K, E, \chi, \lambda, s) e^{-r_\Omega (s-t)} ds \right] \text{ with } r_\Omega \equiv r^* - (\eta - 1) \left(\phi(i^{(0)}) - \eta \sigma_K^2 / 2 \right) + \varphi, \tag{5.3.9}$$

$\phi \equiv \Phi/K = i - \omega i^2/2 - \delta$ and $i = I/K$. The scaled forcing in (5.3.9) is

$$\begin{aligned} \Gamma(K, E, \chi, \lambda, t) \equiv & \big((1 + \theta_{ET}) \varphi \mathrm{X} \Lambda - \nu_\chi (\bar{\chi} - \chi) \mathrm{X}_\chi \Lambda - \nu_\lambda (\bar{\lambda} - \lambda) \mathrm{X} \Lambda_\lambda \\ & - \frac{1}{2} \sigma_\chi^2 \mathrm{X}_{\chi\chi} \Lambda - \frac{1}{2} \mathrm{X} \Lambda_{\lambda\lambda} \sigma_\lambda^2 - (1 - \eta) \mathrm{X}_\chi \Lambda \rho_{K\chi} \sigma_K \sigma_\chi \\ & - (1 - \eta) \mathrm{X} \Lambda_\lambda \rho_{K\lambda} \sigma_K \sigma_\lambda - \mathrm{X}_\chi \Lambda_\lambda \rho_{\chi\lambda} \sigma_\chi \sigma_\lambda \big) K^{1-\eta} E^{\theta_{ET}} \\ & - \theta_{ET} \mu A^{\frac{1}{\alpha}} \left(\frac{1 - \alpha}{b} \right)^{\frac{1}{\alpha}} \mathrm{X} \Lambda K^{2-\eta} E^{\theta_{ET} - 1} e^{-g^{(0)} t} \\ & - \frac{1}{2} \theta_{ET} (\theta_{ET} - 1) \sigma_E^2 \mathrm{X} \Lambda K^{1-\eta} E^{\theta_{ET} - 2} \\ & - \left((1 - \eta) \theta_{ET} \mathrm{X} \Lambda \rho_{KE} \sigma_K \sigma_E + \mathrm{X}_\chi \Lambda \rho_{E\chi} \sigma_E \sigma_\chi + \mathrm{X} \Lambda_\lambda \rho_{E\lambda} \sigma_E \sigma_\lambda \right) K^{1-\eta} E^{\theta_{ET} - 1}, \end{aligned} \tag{5.3.10}$$

where $\mathrm{X} \equiv \chi^{1+\theta_{ET}}$ and $\Lambda \equiv \lambda^{1+\theta_\lambda}$. □

The optimal risk-adjusted SCC in Result 1 is proportional to world GDP and depends directly on the stock of atmospheric carbon through the function $\Theta(E, \chi, \lambda)$. It depends on preferences (ρ, γ and η), geophysical parameters (μ, φ and ν_χ), and the properties of the stochastic processes driving the shocks to GDP, the carbon stock, climate sensitivity and damages. It is evident from Result 1 that the optimal risk-adjusted SCC does not depend directly on the share of fossil fuel in value added or the cost of fossil fuel because of the Cobb-Douglas nature of the production function (5.2.7). Neither does it depend on adjustment costs for physical capital or the depreciation rate of physical capital. It does depend on the growth-corrected return on capital r^* (5.3.8), which to leading-order can be approximated by its value in the absence of climate policy ($P = 0$). Similarly, the investment rate and growth rate of GDP can, to leading-order, be approximated by their values in the absence of climate policy (cf. (D.2.9)):

$$i^{(0)} = Y_K|_{P=0} - q^{(0)} \left(\rho + (\gamma - 1) \left(\phi(i^{(0)}) - \frac{1}{2} \eta \sigma_K^2 \right) \right) \tag{5.3.11}$$

with $Y_K|_{P=0} = \alpha A(E, \chi, \lambda)^{\frac{1}{\alpha}} ((1 - \alpha)/b)^{\frac{1-\alpha}{\alpha}}$ and $g^{(0)} = i^{(0)} - \omega \left(i^{(0)} \right)^2 /2 - \delta \equiv \phi(i^{(0)})$. These follow from the Keynes-Ramsey rule and the capital accumulation equation and subsequently give the price of capital, Tobin's q, as $q(i) = 1/\phi'(i)$. The expected return on investment $r^{(0)}$ equals the sum of the risk-free rate, denoted by $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1 + \gamma) \eta \sigma_K^2 /2$, and the risk premium $\eta \sigma_K^2$.

Result 1 indicates that the absolute error in our expression for the optimal risk-adjusted SCC is $O(\epsilon^2)$ and that the error as fraction of the SCC (itself an $O(\epsilon)$ quantity) is $O(\epsilon)$. Consistently, we ignore the slow dependence of the discount rate on the atmospheric carbon stock, via the marginal productivity of capital, when evaluating the discounting integral in (5.3.9). As $\epsilon \to 0$, the optimal SCC in Result 1 becomes exact. Generally, a closed-form solution to the time integral and the expectations operator over the four stochastic states in (5.3.9) is unavailable, so that Result 1 must be evaluated numerically. This requires five-dimensional numerical integration over the probability space corresponding to the four states and with respect to time.[21] Although such high-dimensional numerical integration is less challenging and computationally demanding than numerical solution of the partial differential equations describing the value function, we can still make considerable analytical headway while introducing only minimal quantitative errors. To do so, we consider only the leading-order effects of uncertainty in Section 5.3. In Section 5.6.4 we demonstrate the numerical accuracy of the resulting tractable rules by comparing them with the numerically exact evaluation of Result 1.

5.4 The optimal risk-adjusted SCC: leading-order effects of uncertainty

To obtain closed-form solutions for the risk-adjusted SCC described by the multi-dimensional integral (5.3.9) in Result 1, we make two additional assumptions. First, we only consider up to leading-order terms in the climatic and damage uncertainties σ_χ^2 and σ_λ^2 and their covariance terms, including with the capital stock (assumption I). Second, the stochastic climate sensitivity and climate damage parameters are initially at their equilibrium values ($\hat{\chi}_0 \equiv \chi_0/\bar{\chi} = 1$ and $\hat{\lambda}_0 \equiv \lambda_0/\bar{\lambda} = 1$) (assumption II). Appendix D.5 implements these assumptions to Result 1 to give Result 2′ in Appendix D.1. Here we present the case with proportional reduced-form damages, normally distributed damage uncertainty and no carbon stock volatility.

Result 2: If $\theta_{ET} = 0$, $\theta_\lambda = 0$ and $\sigma_E = 0$, the leading-order optimal SCC is

$$P = \frac{\mu\, \Theta\, Y|_{P=0}}{r^\star}\left(1 + \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{(\sigma_\chi/\bar{\chi})^2}{r^\star + 2\nu_\chi} + \Delta_{\mathrm{CK}} + \Delta_{\mathrm{CC}}\right) \tag{5.4.1}$$

with

$$\Delta_{\mathrm{CK}} = -(\eta - 1)\sigma_K\left((1+\theta_{\chi T})\frac{\rho_{K\chi}\sigma_\chi/\bar{\chi}}{r^\star + \nu_\chi} + \frac{\rho_{K\lambda}\sigma_\lambda/\bar{\lambda}}{r^\star + \nu_\lambda}\right), \tag{5.4.2}$$

$$\Delta_{\mathrm{CC}} = (1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\bar{\chi}\bar{\lambda}}{r^\star + \nu_\chi + \nu_\lambda} \tag{5.4.3}$$

[21] If the processes are independent, the integrals over the probability space of states can be evaluated independently.

and the discount rate corrected for economic growth, economic growth uncertainty and atmospheric decay is $r^\star = \rho + (\gamma - 1)(g^{(0)} - (1/2)\eta\sigma_K^2) + \varphi$. □

Result 2′ in Appendix D.1 allows for convex damages, skewed damage uncertainty and carbon stock volatility. This leaves the structure of (5.4.1) intact, but includes correction factors that modify the magnitude of terms, but not the interpretation.

5.4.1 The optimal SCC in the absence of economic and climate uncertainty

Without uncertainty, Result 2 gives $P = \mu\Theta(E)\, Y|_{P=0}/r^\star$ with $r^\star = \rho + (\gamma-1)g^{(0)} + \varphi$ for the deterministic optimal SCC. This expression has the same geophysical (μ and φ), economic (Y and g), damage (Θ) and ethical (ρ and γ) determinants as those found in macro models of growth and climate change with Ramsey instead of AK growth. More patience (lower ρ), wealthier future generations (higher $g^{(0)}$ for $\gamma < 1$), and lower intergenerational inequality aversion (lower γ) curb the discount rate and push up the SCC. Rising affluence (higher $g^{(0)}$) pushes up the discount rate, especially if intergenerational inequality aversion is large, and thus reduces the appetite of current generations for ambitious climate policy (the $+\gamma g^{(0)}$ term in $r^\star$). Also, with damages proportional to GDP, rising affluence implies a higher growth of damages and a lower growth-corrected discount rate (the $-g^{(0)}$ term in $r^\star$), which increases the optimal SCC. Higher economic activity (Y) and a higher flow damage coefficient (Θ) also push up the SCC. A smaller fraction of emissions that goes into the atmosphere (smaller μ) and faster rate of decay of atmospheric carbon (higher φ) depress the SCC.

5.4.2 Economic growth uncertainty and the climate beta

Including economic, but not climatic uncertainty, Result 2 gives $P = \mu\ \Theta(E)\, Y|_{P=0}/r^\star$ but now with the risk-adjusted discount rate $r^\star = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2/2) + \varphi$. The estimate of future economic growth is thus reduced to take account of its uncertain nature with risk aversion η. If the rising-affluence dominates the growing-damages effect, growth uncertainty depresses the discount rate and pushes up the risk-adjusted SCC. We decompose the effects on the risk-adjusted discount rate as follows:

$$r^\star = \underbrace{\rho}_{\text{time impatience}} + \underbrace{\gamma g^{(0)}}_{\text{rising affluence}} - \underbrace{g^{(0)}}_{\text{growing damages}} \underbrace{-\frac{1}{2}(1+\gamma)\eta\sigma_K^2}_{\text{prudence}} + \underbrace{\eta\sigma_K^2}_{\text{insurance}} + \underbrace{\varphi}_{\text{decay atmospheric carbon}}. \quad (5.4.4)$$

The first three terms were discussed in Section 5.4.1. The prudence term is proportional to the coefficients of relative prudence CRP $= 1 + \gamma$, and risk aversion η, and to economic growth uncertainty (cf. Leland 1968, Kimball 1990). The insurance term stems from the perfect correlation between damages and GDP, because damages in our model are proportional to GDP. The insurance term acts to increase the optimal discount and reduce the optimal risk-adjusted SCC, reflecting that positive shocks to damages are associated with positive shocks to GDP and thus less harmful to welfare. This corresponds to a "built-in" climate beta of one.[22] For $\gamma > 1$, the prudence term is dominant, the optimal discount rate reduces, and the optimal risk-adjusted carbon price increases with growth uncertainty, and vice-versa for $\gamma < 1$. If utility is logarithmic as in Golosov et al. (2014), $\gamma = \eta = 1$ and $r^\star = \rho + \varphi$, so that uncertainty about the rate of economic growth does not have any impact on the risk-adjusted SCC (cf. Golosov et al. 2014).

If damages are additive and do not rise in proportion to GDP, the correction to the discount rate only consists of the prudence term. More generally, if $0 \leq \theta_D \leq 1$ denotes the elasticity of damages with respect to GDP, the growth-corrected, risk-adjusted discount rate becomes (Jensen and Traeger 2016):

$$r^\star = \rho + (\gamma - \theta_D)g^{(0)} - \frac{1}{2}(1 + \gamma - 2\theta_D)\eta\sigma_K^2 + \varphi. \tag{5.4.5}$$

If the elasticity of damages with respect to GDP is $\theta_D = 1/2$, the effect of growing damages is halved for $\gamma = 1$ (cf. $-\theta_D g^{(0)}$ in (5.4.5)). A smaller elasticity of damages with respect to GDP acts to decrease the insurance term and thus induces a lower risk-adjusted discount rate and higher carbon price. Whether GDP uncertainty increases the carbon price depends on whether the coefficient of relative prudence exceeds twice the elasticity of damages with respect to GDP: CRP $= 1 + \gamma > 2\theta_D$ (Jensen and Traeger 2016, their equation (9)). With multiplicative damages as in Results 1 and 2 ($\theta_D = 1$), this condition reduces to $\gamma > 1$.

We have abstracted from long-run risk in economic growth (Bansal and Yaron 2004).[23] It has been shown numerically that including this long-run risk pushes up the optimal risk-adjusted SCC by a factor 2 or 3 if aversion to risk exceeds aversion to intertemporal fluctuations (Bansal et al. 2016).

5.4.3 Climate and damage uncertainties

The *climate sensitivity risk correction* $(1/2)\theta_{\chi T}(1+\theta_{\chi T})(\sigma_\chi/\bar{\chi})^2/(r^\star+2\nu_\chi)$ in (5.4.1) depends on $\theta_{\chi T} \equiv \theta_\chi + \theta_T + \theta_\chi\theta_T$, which combines the generally convex dependence of temperature on the normally distributed climate sensitivity parameter ($\theta_\chi > 0$, cf. (5.2.5)) and the generally convex dependence of damages on temperature ($\theta_T > 0$). The former captures the (positive) skewness of the (equilibrium)

[22] Dietz et al. (2018) use Monte Carlo simulations of DICE (Nordhaus 2008) and find that the climate beta is close to one if damages are proportional to GDP, but closer to zero if damages are additive. Our Section 5.4.4 analyzes correlated risks and climate betas more generally.

[23] Epstein et al. (2014) argue that long-run risk and a preference of early resolution of uncertainty implies that the timing premium needed to calibrate asset returns is implausibly high (20-30%/year).

climate sensitivity distribution. The climate sensitivity uncertainty correction is thus positive and larger for a more convex damage function, a more (positively) skewed climate sensitivity distribution, a lower rate of mean reversion, greater uncertainty of climate sensitivity, and if the growth-corrected discount rate is smaller (higher θ_T, θ_χ, lower ν_χ, higher σ_χ and lower $r^\star$). There is no corresponding damage uncertainty correction in Result 2, which is consequence of damage uncertainty being normally distributed (i.e. $\theta_\lambda = 0$).

5.4.4 Climate betas: correlated climate, damage and economic growth risks

The term in Result 2 correcting for correlations between climate and damage risks, on the one hand, and economic risks, on the other hand, can be rewritten as

$$\Delta_{CK} = -(\eta - 1)\sigma_K^2 \left(\left(1 + \theta_{\chi T}\right) \frac{\beta_{K\chi}}{r^\star + \nu_\chi} + \frac{\beta_{K\lambda}}{r^\star + \nu_\lambda} \right), \tag{5.4.6}$$

where $\beta_{K\chi} \equiv \rho_{K\chi}\sigma_\chi/\bar{\chi}\sigma_K$ and $\beta_{K\lambda} \equiv \rho_{K\lambda}\sigma_\lambda/\bar{\lambda}\sigma_K$ denote the climate-sensitivity and climate-damage beta, respectively. These climate betas measure the normalized correlation with shocks to the rate of economic growth in direct analogy with the definition of beta in asset pricing theory (e.g. Lucas 1978, Breeden 1979).[24] The sign of (5.4.6) depends on whether relative risk aversion η exceeds one or not, i.e. on whether the *risk-insurance* effect dominates the *risk-exposure* effect or not (cf. Lemoine 2017). If climate sensitivity and economic growth are positively correlated ($\beta_{K\chi} > 0$), the risk-insurance effect pushes down the risk-adjusted SCC, and more so if relative risk aversion is high, climate sensitivity displays less mean reversion, and the climate-sensitivity beta is large (high η, low ν_χ high $\beta_{K\chi}$). Furthermore, the factor $(1 + \theta_{\chi T})$ reflects the increase in the (co-) variance resulting from the power-law dependence of damages on the climate sensitivity parameter χ. The risk-insurance effect is greater for a more convex damage functions (high θ_T) and a more skew climate sensitivity (high θ_χ), but remains non-zero even for a symmetric climate sensitivity distribution and linear damages ($\theta_\chi = \theta_T = 0$). The risk-exposure effect acts in the opposite direction: if the climate sensitivity parameter χ (and temperature) is high and $\beta_{K\chi} > 0$, the adverse effects (on GDP) are amplified due to the multiplicative nature of damages, requiring a rise in the carbon price. The risk-insurance effect dominates the risk-exposure effect if $\eta > 1$.

Finally, the term $\Delta_{\text{CC}} = (1+\theta_{\chi T})\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\bar{\chi}\bar{\lambda}(r^\star + \nu_\chi + \nu_\lambda)$ in Result 2 captures the correlation between climate sensitivity and damage uncertainty: risk aversion η plays no role as there is no insurance possibility via the economic growth channel. If climate sensitivity uncertainty and damage uncertainty are positively correlated, this term is positive and thus the risk-adjusted SCC is pushed upwards.

[24]Consistent with our perturbation scheme, the volatility of total GDP is given to leading order by the volatility of the capital stock neglecting the effect of climate damages and thus the carbon stock, climate sensitivity and damage uncertainties.

5.4.5 Special case: logarithmic preferences

With logarithmic preferences and proportional reduced-form damages, $\gamma = \eta = 1$ and $\theta_{ET} = 0$ (cf. Golosov et al. 2014), we have $\Delta_{\mathrm{CK}} = 0$ and Result 2 becomes

$$P = \frac{\mu\,\Theta\,Y|_{P=0}}{\rho+\varphi}\left(1 + (1+\theta_{\chi T})(\sigma_\chi/\bar{\chi})\left[\frac{1}{2}\theta_{\chi T}\frac{(\sigma_\chi/\bar{\chi})}{\rho+\varphi+2\nu_\chi} + \frac{\rho_{\chi\lambda}(\sigma_\lambda/\bar{\lambda})}{\rho+\varphi+\nu_\chi+\nu_\lambda}\right]\right). \tag{5.4.7}$$

Hence, economic growth uncertainty and the climate betas do not affect the optimal risk-adjusted SCC, but climatic uncertainty and correlated risks for climate sensitivity and damage uncertainty do.

5.5 Calibration

Table 5.1 summarizes the details of our calibration with further details in Appendix D.6. To calibrate the non-climatic part of our model to match historical asset returns, we follow Pindyck and Wang (2013) but abstract from catastrophic shocks to economic growth (see Appendices D.6.1 and D.6.2). We note that carbon stock volatility is extremely small.[25] By setting $\theta_E = -0.5$ and $\theta_T = 1$, we have proportional reduced-form damages with $\theta_{ET} = 0$ (see Appendices D.6.3 and D.6.4). By setting $\theta_T = 1.5$, we have convex reduced-form damages with $\theta_{ET} = 0.25$. We let damage uncertainty be normally distributed, so $\theta_\lambda = 0$, with mean μ_λ and standard deviation Σ_λ calibrated to the estimates surveyed in Tol (2009) and associate these with the steady-state distribution, so $\mu_\lambda = \bar{\lambda}$ and $\Sigma_\lambda^2 = \sigma_\lambda^2/2\nu_\lambda$ (see Appendix D.6.5), setting $\nu_\lambda = 20\%$/year. The flow damage coefficient $\Theta \equiv D_E/(1-D)$ with μ_χ and θ_χ set to our base case values is approximately constant at 2.6% GDP/TtC for proportional damages but starts at 3.2% of GDP/TtC and then rises with global warming for convex damages. For comparison, Golosov et al. (2014) have $\Theta = 3.64\%$ GDP/TtC, which includes a markup for tipping risk.

The equilibrium climate sensitivity (ECS) is defined as the equilibrium change in annual mean global temperature following a doubling of the atmospheric carbon stock relative to pre-industrial levels. From (5.2.5) the climate sensitivity is $T_2 \equiv T(E = E_{PI}, \chi) = \chi^{1+\theta_\chi}$, where χ is normally distributed with mean μ_χ and standard deviation Σ_χ, and θ_χ is chosen to match the skewness of the climate sensitivity T_2. We fit the distribution of T_2 to a range of estimates for the ECS, thereby getting close to the thin-tailed Gamma distribution of Pindyck (2012).

[25] Using the same dataset, but considering a geometric Brownian Motion for the atmospheric carbon concentrations above pre-industrial level instead of the arithmetic Brownian Motion considered here, Hambel et al. (2017) find a much larger volatility of 0.78%/year$^{1/2}$. Estimating this volatility, we find 1.4, 0.5 and 0.2%/year$^{1/2}$ for the periods 1800-2004, 1900-2004 and 1959-2004. This large variation of volatility with time suggest that historical volatility in the atmospheric carbon concentrations is better described by an arithmetic Brownian Motion, as in (5.2.4).

Table 5.1: Summary of the base case calibration.

Rate of impatience	$\rho = 5.75\%$/year
Intertemporal substitution (inverse of intergenerational inequality aversion)	$EIS = 1/IIA = 1/\gamma = 0.67$
Attitudes to risk	$CRRA = \eta = 4.32$
World economy	$A^* = 0.113$/year, GDP = \$75 trillion/year $g^{(0)} = 2.0\%$/year
Investment and adjustment cost	$I_0^{(0)}/Y_0^{(0)} = 24.3\%$, $i^{(0)} = 2.75\%$/year $\delta = 0.28\%$/year, $\omega = 12.3$ year
Asset volatility and returns	$\sigma_K = 12.13\%/\text{year}^{1/2}$ $r^{(0)} = 7.16\%$/year, $r_{\text{rf}}^{(0)} = 0.80\%$/year $r^{(0)} - r_{\text{rf}}^{(0)} = \eta\sigma_K^2 = 6.36\%$/year
Share of fossil fuel in value added	$1 - \alpha = 6.6\%$
Production cost of fossil fuel	$b = \$5.4 \times 10^2$/tC
Pre-industrial and 2015 ($t = 0$) carbon stocks	$S_{PI} = 596$ GtC, $S_0 = 854$ GtC, $E_0 = 258$ GtC,
Stochastic carbon stock dynamics	$\mu = 1.0$, $\varphi = 0.66\%$/year, $\sigma_E = 0.31$ GtC/year$^{1/2}$
Concavity temperature function	$\theta_E = -0.5$
Convexity and mean reversion damages	$\theta_\lambda = 0$, $\nu_\lambda = 20\%$/year
Mean and standard deviation of damage uncertainty	Proportional damages: $\theta_T = 1$, $\mu_\lambda = 2.2 \times 10^{-3}$, $\Sigma_\lambda = 1.6 \times 10^{-3}$ Convex damages: $\theta_T = 1.5$, $\mu_\lambda = 1.6 \times 10^{-3}$, $\Sigma_\lambda = 1.0 \times 10^{-3}$
Flow impact global warming damages	Proportional damages: $\Theta_0 = 2.63\%$ GDP/TtC Convex damages: $\Theta_0 = 3.16\%$ GDP/TtC
Distribution of the ECS	$\mu_\chi = 1.9$, $\Sigma_\chi = 0.95$, $\sigma_\chi = 11\%/\text{year}^{1/2}$, $\nu_\chi = 0.66\%$/year, $\theta_\chi = 0.59$, $\theta_{\chi T} = 2.2$ and 3.0 for proportional and convex damages, respectively
Distribution of the TCR	$\mu_\chi = 1.75$, $\Sigma_\chi = 0.38$, $\sigma_\chi = 4.5\%$ /year$^{1/2}$, $\nu_\chi \to 0$, $\theta_\chi = 0$, $\theta_{\chi T} = 1.0$ and 1.5 for proportional and convex damages

This yields $E[T_2] = 3.0°\text{C}$, $\text{var}[T_2] = 4.5°\text{C}^2$ and $\text{skew}[T_2] = 10°\text{C}^3$, indicating a right-skewed equilibrium distribution. We estimate the rate at which this skew equilibrium distribution is reached at $\nu_\chi = 0.66\%$/year, capturing that the ECS and its associated skewness occurs on time scales of a few centuries. For comparison, we also fit the non-skew transient climate response (TCR), which is defined as the change in annual mean global temperature at the time of doubling following a linear increase in the carbon stock IPCC (2013). Matching information from Figure 10.20 and Chapter 10 of IPCC (2013), we obtain $E[T_2] = 1.75°\text{C}$,

Table 5.2: Two ways of calibrating climate sensitivity.

	ECS (steady state)	**TCR (after 70 years)**
$E[T_2]$	3.0°C	1.75°C
$\mathrm{var}[T_2]$	4.5°C^2	0.15°C^2
$\mathrm{skew}[T_2]$	10°C^3	0
$\mathrm{skew}^*[T_2]$	1.0	0

$\mathrm{var}[T_2] = 0.15°\mathrm{C}^2$ and $\mathrm{skew}[T_2] = 0$, indicating a mean TCR of 1.75°C and a normal (non-skew) distribution. Table 5.2 compares our ECS and TCR calibrations. The skewness, despite being a long-run feature only, is the most important driver of the risk-adjusted SCC, and we adopt the ECS calibration in our base case (Appendix D.6.6).

5.6 Quantification of effects of economic and climatic risks on optimal SCC

Table 5.3 gives the deterministic and the risk-adjusted SCC[26] for the two ways of calibrating climate sensitivity. The effect of atmospheric carbon stock uncertainty is identically zero or numerically negligible with, respectively, proportional and convex reduced-form damages. The markup for climate sensitivity uncertainty on the deterministic optimal carbon price varies from 2% for the TCR calibration and 22% for the ECS calibration for proportional damages or 3% (TCR) and 37% (ECS) for the convex variant. The ECS calibration leads to a larger upward adjustment of the price due to the marked skewness of the distribution in the ECS, which is not present in the distribution of the TCR.[27] The deterministic SCC for the TCR is also much lower due to the lower temperature rise associated with this climate sensitivity and the lower curvature of the flow damage coefficient.[28]

Although the damage specification of Nordhaus (2008) is associated with a curvature that is approximately constant at $\theta_T = 1$, corresponding to our proportional damages case ($\theta_{ET} = 0$), the curvature of the specification by Ackerman and Stanton (2012) increases rapidly after approximately 1°C of warming to a value of 4. The effect of the degree of convexity θ_T is considerable, as illustrated by the increase of the markup from 22% to 37% between $\theta_T = 1.0$ and $\theta_T = 1.5$. An even greater convexity of the generally poorly understood and ad-hoc damage function is not

[26] We use Result 2′ and refer to the definitions of $r^\star$ and other variables to Appendix D.1.

[27] If we calibrate the climate sensitivity to the transient response to cumulative emissions (see Appendix D.6.6), we obtain for the case of proportional damages a deterministic SCC of \$4.13/tCO_2 and markups for economic and climatic risk of 30% and 6%, respectively, thus leading to a risk-adjusted carbon price of 5.71/tCO_2.

[28] For the TCR calibration, Θ_0 is 1.13% (proportional) or 1.11% of GDP/TtC (convex damages).

Table 5.3: The effects of risk on the optimal SCC.

Damages	**Proportional**			**Convex**		
Carbon price ($/tCO$_2$)	**Base case**	**TCR**	$\nu_\chi = \infty$	**Base case**	**TCR**	$\nu_\chi = \infty$
Deterministic carbon price	7.27	3.13	7.27	9.38	3.28	9.38
Due to economic growth uncertainty	1.99	0.86	1.99	2.11	0.74	2.11
Due to carbon stock uncertainty	0.00	0.00	0.00	0.00	0.00	0.00
Due to climate sensitivity uncertainty	1.56	0.05	8.45	3.46	0.09	17.93
Due to climate damage uncertainty	0	0	0	0	0	0
Total risk-adjusted carbon price	10.81	4.04	17.70	14.96	4.11	29.43
(Total risk markup)	(49%)	(29%)	(144%)	(59%)	(25%)	(214%)
(Climate risk markup)	(22%)	(2%)	(116%)	(37%)	(3%)	(191%)

inconceivable.

The magnitude of the markup for the uncertain nature of the skew equilibrium climate sensitivity is determined by the time scale over which the equilibrium is reached. In our base case calibration, we have an e-folding time of $1/\nu_\chi = 1.5 \times 10^2$ years, whereas Ricke and Caldeira (2014) argue that temperature rises very quickly, on the time scale of a decade, after a (small) carbon impulse. An upper limit to the climate sensitivity uncertainty correction corresponds to the limit $\nu_\chi \to \infty$, in which the skew equilibrium sensitivity can be thought to arrive instantaneously following emissions. This gives total risk-adjusted carbon prices of \$17.70 and \$29.43 per ton of CO_2 and climate risk markups of 116% and 191%, for proportional and convex damages, respectively. Although climate damages are subject to considerable uncertainty, there seems no evidence for non-zero skewness, resulting in no correction to the optimal carbon price. The main effect of climate damage uncertainty is through correlation with GDP, as discussed in Section 5.6.2.

5.6.1 Economic determinants of the optimal carbon price

Table 5.4 shows the effects of preferences and economic growth on the risk-adjusted SCC and its stochastic drivers. We use the more convex damage variant and the ECS calibration as a base. We recall that the optimal discount rate in the absence of stochastic climate corrections is $r^\star = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2/2) + (1 + \theta_{ET})\varphi$ (from Result 2′ in Appendix D.1). It is evident that $CRRA = \eta$ has a large downward effect on the discount rate and upward effect on the SCC if economic uncertainty is high (and $\gamma > 1$). If $CRRA = 10$ instead of 4.32 (base case), we obtain a much

Table 5.4: Effects of *CRRA*, *IIA*, growth uncertainty and correlated risks.

	ECS calibration with convex damages						
Carbon price ($/t$CO_2$)	**Base case**	***CRRA* = 10**	***IIA* = 3**	**Annual σ_K = 1.5%**	**Climate Beta $\theta_D = 0.5$**	**Climate Beta $\rho_{K\lambda} = 0.5$**	**$\rho_{\chi\lambda}$ = 0.5**
Deterministic	9.38	9.38	6.57	9.38	9.38	9.38	9.38
Due to economic growth uncertainty	2.11	7.46	9.77	0.02	14.67	2.11	2.11
Due to climate sensitivity uncertainty	3.46	7.30	6.53	2.33	13.13	3.46	3.46
Due to climate damage uncertainty	0	0	0	0	0	−3.73	2.04
Total risk-adjusted	14.96	24.14	22.87	11.74	37.18	11.23	17.00
(Total risk markup)	(59%)	(157%)	(248%)	(25%)	(296%)	(20%)	(81%)
(Climate risk markup)	(37%)	(78%)	(100%)	(25%)	(140%)	(−3%)	(59%)

higher economic uncertainty correction of \$7.46 per ton of CO_2. The correction for climate uncertainty as a share of the deterministic SCC more than doubles. The total risk markup is much higher: 157% instead of 59% (second column). A higher aversion to intergenerational inequality, e.g. an *IIA* of 3 instead of 1.5, pushes down the deterministic SCC from \$9.38 to \$6.57 per ton of CO_2, but pushes up the correction for economic uncertainty from \$2.11 to \$9.77 per ton of CO_2 (third column), but less so if economic uncertainty is lower (fourth column). For an *IIA* of 3, the correction for climate sensitivity uncertainty is pushed up from \$3.46 to \$6.53 per ton of CO_2, the risk markup to 248%, and the risk-adjusted SCC becomes \$22.87 per ton of CO_2.

Our calibration is based on historical asset returns. If instead, we calibrate based on historical GDP, a much smaller annual volatility of 1.5% is appropriate (cf. Hambel et al. 2017). Hence, the correction for economic growth uncertainty shrinks from \$2.11 to a mere \$0.02 per ton of CO_2. As a result of the smaller downward correction of the discount rate, the correction to allow for the risk of climate sensitivity uncertainty is cut from \$3.46 to \$2.33 per ton of CO_2 (fourth column) due to an increase in the growth-corrected discount rate $r^\star$. The risk-adjusted SCC drops from \$14.96 to \$11.74 per ton of CO_2, corresponding to a total risk markup of only 25% instead of 59%.

The coefficient of relative risk aversion *CRRA* has a large effect in our base case calibration, which is due to the large volatility of asset returns. For the much smaller volatility of historical GDP of 1.5%, the correction for economic growth uncertainty only increases from \$0.02 to \$0.05 per ton of CO_2 as *CRRA* is increased to 10 (not shown in Table 5.4). This accords with Crost and Traeger (2013), Ackerman et al. (2013) and Hambel et al. (2017), who all use a small value for economic uncertainty and find that *CRRA* only has a small and that *IIA* has a large effect on the risk-adjusted SCC (which follows from our growth-corrected discount rate $r^\star = \rho + (\gamma - 1)(g^{(0)} - \eta\sigma_K^2/2) + (1 + \theta_{ET})\varphi$).

5.6.2 Climate betas

If damages are not proportional to world GDP, but instead the elasticity of damages with respect to world GDP is only a half, we can compute the effect of setting the "climate beta" θ_D to 0.5 instead of 1 using the ad-hoc modification (5.4.5). Since damage shocks are no longer automatically insured against by their direct proportionality with GDP, the optimal growth-corrected discount rate $r^\star$ drops, as reflected by a much larger correction for uncertain economic growth (we keep the deterministic price fixed). Because of the lower discount rate, the climate sensitivity uncertainty correction rises significantly. The risk-adjusted SCC thus increases to \$37.18 per tCO_2 corresponding to a very large risk markup of 296% (Table 5.4, fifth column). A more direct approach to the climate beta is to consider the correlation structure of the uncertain processes themselves. For example, if we set $\rho_{K\lambda} = 0.5$ to capture that adverse damage shocks in the future are more likely if economic growth is high, the SCC must be adjusted downwards by \$3.73 per tCO_2 due to the dominance of the insurance effect for *CRRA* > 1 (Table 5.4, sixth column). Alternatively, if we set $\rho_{\chi\lambda} = 0.5$ to capture that an adverse future climate sensitivity shock might be associated with an adverse future damage shock, the SCC is pushed up by \$2.04 per tCO_2 (seventh column). Given the very small carbon stock uncertainty, the effects of $\rho_{EK}, \rho_{E\chi}$ and $\rho_{E\lambda}$ are negligible.

5.6.3 Comparison with other calibrations

Table 5.5 compares our base case results to common alternatives, which rely on ethical arguments to use much lower discount rates than derived from asset market returns (e.g. Gollier 2018).[29] In contrast to the base case, we assume a low economic growth volatility of 1.5%/year$^{1/2}$, based on GDP instead of asset return volatility, for all alternative calibrations in Table 5 except those in bold. We adopt the calibration based on the ECS and proportional damages for all alternative calibrations except for Stern. As all corrections for damage and carbon stock uncertainty are zero or negligible, we do not show these rows. Golosov et al. (2014) (GHKT) adopt logarithmic utility, *IIA* = *CRRA* = 1, and ρ = 1.5% per year. From (5.3.8) then, neither the expected rate of growth nor the uncertainty of the future rate of economic growth influences the optimal SCC. The growth-corrected

[29]To analyze this properly, the government should maximize expected welfare using low ethically motivated discount rates, subject to the constraints of the decentralized market economy calibrated to higher asset returns. The optimal carbon price will then typically fall short of the social cost of carbon (Belfiori 2017, Barrage 2018).

Table 5.5: Comparison with other calibrations.

Base case $\theta_T = 1$	**Base case $\theta_T = 1.5$**	**GHKT $\theta_T = 1$**	**Gollier $\theta_T = 1$**	**Nordhaus $\theta_T = 1$**	**Stern $\theta_T = 1$**	**Stern $\theta_T = 1$**	**Stern $\theta_T = 1.5$**	**Stern $\theta_T = 1.5$**
Deterministic								
7.27	**9.38**	24.91	20.23	17.58	32.41	**32.41**	46.97	**46.97**
Due to economic growth uncertainty								
1.99	**2.11**	0	0.17	0.04	0.14	**13.19**	0.13	**6.52**
Due to climate sensitivity uncertainty								
1.56	**3.46**	8.64	6.22	4.86	13.21	**22.01**	34.66	**44.33**
Total risk-adjusted								
10.81	**14.96**	33.55	26.62	22.49	45.77	**67.62**	81.76	**97.82**
(Total risk markup)								
(49%)	**(59%)**	(35%)	(32%)	(28%)	(41%)	**(109%)**	(74%)	**(108%)**
(Climate risk markup)								
(22%)	**(37%)**	(35%)	(31%)	(28%)	(41%)	**(68%)**	(74%)	**(94%)**

The base case has proportional damages, ECS calibration and annual economic volatility of 12.1%. GHKT is based on Golosov et al. (2014) and has $IIA = CRRA = 1$ and $\rho = 1.5\%$/year. Gollier (2012) has $IIA = CRRA = 2$, $\rho = 0$. Nordhaus (2008) has $IIA = CRRA = 1.45$ and $\rho = 1.5\%$/year. Stern (2007) has $IIA = CRRA = 1.45$ and $\rho = 0.1\%$/year. The last two columns have convex damages with $\theta_T = 1.5$. The columns in bold use an annual asset volatility of 12.1%.

discount rate $r^\star$ is only 2.16% instead of 5.82% per year, and thus the deterministic SCC is almost fourfold under GHKT. Since the discount rate is so much lower, the adjustments for climate risk are 5-6 times higher, but less so if expressed as share of the deterministic price. The risk-adjusted SCC under GHKT is more than three times as high as in our base (\$33.55 instead of \$10.81 per tCO_2).

Gollier (2012) focuses on the risk-adjusted discount rate. He suggests using $IIA = CRRA = 2$ and $\rho = 0$, so that $r^\star$ becomes 2.64% per year. As this is more than for GHKT, but less than for our base case, the deterministic SCC is higher than for the base but lower than for GHKT. Intergenerational inequality aversion (IIA) now exceeds one. Thus, there is a positive adjustment for the carbon price to take account of uncertain economic growth, but it is small given that volatility is calculated from GDP instead of asset returns. The adjustment for climate sensitivity uncertainty is approximately 4 times higher than for the base case due to the low discount rate. The risk-adjusted SCC is much higher, but less than under GHKT.

The integrated assessment model DICE developed by Nordhaus (2008) has $IIA = CRRA = 1.45$, $\rho = 1.5\%$ per year and thus a growth-corrected discount rate $r^\star$ of 3.05% per year ($\theta_T = 1$). As a result, the deterministic SCC, the correction for climate sensitivity risk and the fully risk-adjusted SCC are lower than under Gollier and GHKT, but higher than for the base case. The final four columns change the discount rate to a much lower value, as may be justified on ethical grounds (cf. Stern 2007). Setting $\rho = 0.1\%$ per year and keeping $IIA = CRRA = 1.45$ gives a risk-adjusted discount rate $r^\star$ of 1.65% per year, which gives a risk-adjusted SCC of \$45.77 per tCO_2, as illustrated in the first of these four columns. Focusing on the Stern variant, the next column in bold indicates that, if economic volatility is calculated from asset returns instead of GDP, both the correction for economic growth uncertainty and, due to the lower risk-adjusted discount rate, the correction for climate sensitivity uncertainty rise substantially. As a result, the risk-adjusted SCC is pushed up from \$45.77 to \$67.62 per tCO_2. The last two columns show a more realistic version of the Stern variant, namely with convex damages. These boost the deterministic SCC from \$32.41 to \$46.97 per tCO_2. And the risk-adjusted SCC from \$45.77 to \$81.76 per tCO_2 if economic volatility is calibrated from GDP and from \$67.62 to \$97.82 per tCO_2 if it is calibrated based on asset returns.

5.6.4 Accuracy of the tractable rule for the optimal risk-adjusted carbon price

To assess the accuracy of the approximations made in Result 2 and 2′ used in Tables 3-5, we evaluate Result 1 numerically (see Appendix D.7 for details). For all possible calibrations considered, the error is small (less than 2.6% for convex damages but less than 0.3% for proportional damages). Crucially, the effect of ignoring carbon stock uncertainty arising from uncertain future emissions in Result 2′ (see Appendix D.1) is negligibly small.

5.7 Conclusions

Using asymptotic methods, this chapter has derived a tractable rule for the optimal risk-adjusted SCC under a range of economic and climatic uncertainties allowing for the convexity of global warming damages and the skewness of shocks to the climate sensitivity and global warming damages, and the time scales on which they arise. This gives insight into the ethical determinants and the stochastic economic and geophysical drivers of the optimal carbon price and is a very good approximation to our more fundamental result, which only requires that climate damages are a small percentage of world GDP (say, less than 10%). Our rule offers a powerful analytical complement to insights that could hitherto only be derived from numerical solutions of systems of stochastic differential equations.

With damages proportional to the carbon stock, our optimal SCC is also proportional to world GDP. However, if damages are convex, the proportion rises over time as global warming increases. The rate used to discount marginal damages must be corrected for the various economic, climate and damage risks. The risk-adjusted SCC increases in risk aversion but decreases in intergenerational inequality aversion. The effect of risk aversion is quantitatively much smaller. If the elasticity of damages with respect to world GDP is less than one, the climate beta is less than one and the risk-adjusted SCC is higher.

Taking account of uncertainty in the carbon stock dynamics leads to negligible adjustment of the optimal SCC. Uncertain climate sensitivity does act to increase the SCC significantly, especially if allowance is made for the skewness of the equilibrium climate sensitivity distribution. Crucially, only the equilibrium climate sensitivity is associated with significant skewness, and the role this plays in determining the optimal SCC depends strongly on the time horizon over which this equilibrium is reached. Taking account of the uncertainty about the economic impact of damages does not affect the optimal SCC, unless this distribution is skew, for which we have not found a-priori evidence.

Furthermore, our solutions allow insight into the origins of the overall climate beta by allowing for correlated risks in the economic growth rate, the carbon stock, the climate sensitivity and damages. The risk-adjusted SCC is pushed up if stochastic shocks to the climate sensitivity or the climate damage parameter are negatively correlated with shocks to the future rate of economic growth, provided the degree of risk aversion exceeds one. If shocks to damages are negatively correlated with stochastic shocks to the future rate of economic growth, the corresponding climate beta shows by how much more carbon should be priced. The directions of these effects reverse if the risk-exposure effect dominates the risk-insurance effect, i.e. if the coefficient of relative risk aversion is less than one. These effects do not depend on intergenerational inequality aversion and they do not impact the optimal SCC if risk aversion equals one.

Our quantitative results suggest that with convex damages the markups on the deterministic SCC for economic and climate sensitivity uncertainties are 22% and 37%, respectively, giving a risk-adjusted SCC of \$15/tCO_2. However, if the elasticity of damages from global warming with respect to world GDP is $1/2$ instead of 1, these markups are 156% and 140%, respectively. The SCC thus more than doubles to \$37/tCO_2 as world GDP acts less as insurance. If the correlation coefficient between economic and damage uncertainties is $1/2$, capturing that adverse damage shocks in the future are more likely if economic growth is high, the SCC needs to be adjusted downwards by \$4 per tCO_2. But if the correlation coefficient for climate sensitivity and damage uncertainties is $1/2$, the climatic risk markup rises (as the coefficient of relative risk aversion exceeds 1 in our calibration) from 37% to 59%. If a low ethical instead of a market-based discount rate is used as in Stern (2007), the deterministic SCC rises to \$47/tCO_2, the economic and climatic risk markups are 14% and 94%, respectively, and the risk-adjusted SCC becomes \$98/tCO_2. However, if economic volatility is calibrated to the lower volatility of GDP instead of asset returns, the economic and climatic risk markups are negligibly small and 74%, respectively, and the risk-adjusted SCC is still as high as \$82/tCO_2.

Future investigations should be directed at obtaining robust empirical estimates of the climate betas, a largely uncharted territory. Other areas for future research are to extend our analytical approach to the optimal risk-adjusted price of carbon to include compound Poisson shocks to climate sensitivity (e.g. Hambel et al. 2017) or for richer positive feedback processes in the uptake of atmospheric carbon due to the CO_2 absorption capacity of the oceans declining with temperature (Millar et al. 2016). Each of these extensions would push up the optimal price of carbon, as would the risk of tipping points (e.g. Lemoine and Traeger 2014, 2016, Lontzek et al. 2015, Cai et al. 2016, van der Ploeg and de Zeeuw 2018). Finally, mean reversion in the stochastic process for the rate of economic growth and a downward-sloping term structure with risk aversion exceeding aversion to intertemporal fluctuations (Gollier and Mahul 2017) and compound Poisson shocks to capture catastrophic shocks to total factor productivity
(cf. Bretschger and Vinogradova 2018, Bansal et al. 2016) also push up the optimal risk-adjusted SCC. Future work will employ asymptotic methods to identify tractable leading-order solutions for these circumstances.

Chapter 6

Discussion

This thesis has examined the optimal macroeconomic policy response to the uncertainty associated with natural resource revenues and the uncertainty associated with climate change. For countries endowed with significant reserves of natural resources, the questions are whether to save, spend or invest the resulting income, how much extra to save in the face of uncertainty, and how to allocate these savings in a portfolio of financial assets. For the world as a whole, facing a changing and uncertain future climate, the question is how to put a value on these uncertain changes in order to decide on the strength of action, as measured by a carbon price.[1]

The valuation of the impact of climate change, and perhaps also that of very long-lasting natural resource windfalls, is subject to far longer time scales and far greater uncertainty than almost all other, small-scale and short-horizon, problems in environmental economics, for which at least a certain amount consensus exists about the choice of discount rates in cost-benefit analysis (cf.Pindyck (2013b), Gollier (2012, 2018)). Some of this uncertainty is in the realm of the "unknown unknowns", famously illustrated by Donald Rumsfeld's 2002 quote.[2]

The Ellsberg paradox (Ellsberg 1961) suggests that people's behaviour is different in risky situations when they are given objective probabilities from their behaviour in ambiguous situations when the odds are not known. Modern decision theory has extended the expected utility framework used in this thesis to allow for aversion to ambiguity.[3] Our knowledge of the impacts of climate change and long-

[1] See van der Ploeg and Withagen (2017) and van der Meijden et al. (2017) for recent reviews of the challenges of implementing climate policy.

[2] " *(...) as we know, there are known knowns; there are things we know we know. We also know there are known unknowns; that is to say we know there are some things we do not know. But there are also unknown unknowns the ones we don't know we don't know. And if one looks throughout the history of our country and other free countries, it is the latter category that tend to be the difficult ones.*" The quote is from a response United States Secretary of Defense Donald Rumsfeld gave to a question at a U.S. Department of Defense news briefing on February 12, 2002, available online `http://archive.defense.gov/Transcripts/Transcript.aspx?TranscriptID=2636`

[3] See Epstein and Schneider (2010) for a review in the context of asset prices and, for example,

term natural resource prices may not be of sufficient quality to justify probabilistic beliefs. Expected utility theory, as used throughout this thesis, may therefore not be the right standard of rationality. In the context of scientific uncertainty regarding climate, Millner et al. (2013) have shown that greater ambiguity aversion may lead to more or less abatement of emissions, depending on the details of the model and that this "ambiguity premium" may be large.[4]

In order to still provide answers to the above questions (in the framework of expected utility theory), which are evidently related but have been addressed separately in this thesis, a number of strong assumptions have been made and key elements of the problem ignored. To discuss these limitations, place the contribution made in this thesis in the context of more and less recent developments in the literature and outline potential avenues for future research, I will distinguish the following three themes below: non-resource and non-climatic uncertainty (§6.1), uncertain natural resource reserves and stranded assets (§6.2) and climate uncertainty and catastrophes (§6.3)[5] The focus of this discussion will be on the modelling of uncertainty.

6.1 Non-resource and non-climatic uncertainty

Proceeding sequentially from Chapter 2 to Chapter 5, an increasing number of sources of uncertainty has been taken into account. Whereas Chapters 2 and 3 have considered the uncertainty arising from natural resource prices in isolation,[6] Chapter 4 has considered natural resource price uncertainty in conjunction with the uncertainty of asset returns and Chapter 5 has considered climate uncertainty in conjunction with uncertain economic growth, neither of the uncertainty specifications can explain the well-known equity premium and risk-free rate puzzles (Mehra and Prescott 1985, Weil 1989). Although Chapters 4 and 5 have separated risk aversion and elasticity of intertemporal substitution through recursive preferences (Epstein and Zin 1989, Kreps and Porteus 1978, Duffie and Epstein 1992), such a preference specification can still not fully explain these puzzles. Together, incorporating time-varying long-run growth rates (Bansal and Yaron 2004) and rare macroeconomic disasters (Barro 2006) can probably explain high stock-price volatility and thus the equity premium and risk-free rate puzzles with reasonable values of the coefficients of intertemporal substitution and relative risk aversion

Traeger (2014) for a discussion of discounting under intertemporal risk aversion and ambiguity.

[4]Ambiguity aversion may also be an appropriate response to climate change deniers (van der Ploeg and Rezai 2017).

[5]See also Gillingham et al. (2018) for a recent review of uncertainty modelling in integrated assessment of climate change.

[6]Necessitating relatively arbitrary assumptions about the social discount rate, as discussed in §3.2.2 (see also Frederick et al. (2002) for a review of time discounting and time preference).

(Barro and Ursúa 2012, Wachter 2013).[7]

Uncertain economic growth in Chapters 4 and 5 was assumed to evolve according to a Brownian motion process[8], which gives a constant additive modification to the discount rate due to uncertainty and prudence of the utility function (or a flat "term structure", cf. Hansen and Singleton (1983)). In the long-run risks asset pricing model of Bansal and Yaron (2004),[9] on the other hand, shocks are persistent and continue to affect the growth rate over multiple periods. The shocks themselves display mean reversion, and the resulting long-term volatility is much larger than the short-term volatility and an increasing function of the persistence of shocks. Accordingly, the risk premium arises from dividends paid in the far future. Separately, Rietz (1988) introduced rare disasters into an asset-pricing and argued it helps explain the equity premium puzzle. Barro (2006) provided empirical support for this hypothesis by examining long-term data of real per-capita GDP for many countries and thus including numerous realizations of disaster events.[10][11]

What are the implications of not including these two important developments in modern financial economics in the chapters of this thesis? Evidently, including long-term risk and macroeconomic disasters would allow the model to better capture the stochastic features of the economy, even in the absence of resource and climatic uncertainty, and thus necessitate less arbitrary assumptions about the discount rate. Regarding natural resource revenues in Chapters 2 and 3 and even without including long-run risk and disasters, the precautionary motive of an uncertain oil price will be enhanced if volatility of the rest of the economy is included, as the two will be highly correlated. With exogenous oil rents, there is only a risk-exposure effect and not a risk-insurance effect (in the language of Chapter 5). For the Canadian province of Alberta, the implications of this positive correlation are acutely felt in every economic downturn, and the estimates of optimal precautionary savings in Chapter 3 are thus lower bounds.

In short, correctly capturing the time-varying nature of market risk, resource prices and climate risk and their covariance structure will have important impli-

[7]The concepts time-varying long-run growth rates and rare disasters have both been extended to climatic risk, as discussed in §6.3.

[8]Precisely, the capital stock was described by a geometric Brownian motion, resulting in Brownian motions for the growth rates of GDP and consumption in both the CAPM model of Chapter 4 and the AK growth model of Chapter 5 (at the lowest order in the expansion, i.e. ignoring climate change).

[9]Benzoni et al. (2011) have extended the long-run risk model of Bansal and Yaron (2004) to continuous time. These authors also add jumps processes.

[10]The initial data only included GDP, not consumption, and are subject to flaws in construction, especially at times of disasters such as wars (see Barro and Ursúa (2012) for a review).

[11]Taking an alternative approach, Pindyck and Wang (2013) computed rare disaster characteristics from aggregate asset market data instead of GDP or consumption and estimated the implications of rare disasters for catastrophic risk insurance.

cations for the magnitude of the precautionary motive given the long horizons involved. Indeed, Bansal et al. (2016), using the long-run risk temperature model of Bansal and Ochoa (2011a) find very significant effects of including long-run risk on the social cost of carbon.[12]

6.2 Uncertain natural resource reserves and stranded assets

Although Chapter 4 has considered the effect of stochastic oil prices on optimal extraction, finding that the rate of oil extraction should be faster than predicted by the standard Hotelling rule if oil prices are volatile and positively correlated with financial markets and extraction costs convex, it has not considered the uncertain nature of extractable reserves.[13] It is clear from any historical record of oil or gas reserves (e.g. BP (2017)) that shocks are very significant and generally positive (i.e. discoveries). The question whether to spend or save the proceeds from the extraction of an uncertain reserve cannot be considered separately from the optimal extraction decision. Indeed, changes in oil production can be used to explain part of the observed oil price volatility (Hamilton 2009, Baumeister and Kilian 2016).

What are the implications of not including reserve uncertainty in this thesis? If not only the price, but also quantity of reserves are subject to uncertainty, Chapters 2 and 3 likely underestimate the amount of precautionary saving that is optimal (prudence). More rapid extraction may be optimal in order to reduce the uncertainty in the value of total wealth (risk aversion), which may be balanced by more saving to keep the percentage out of total income consumed unchanged. If the uncertainty in reserve estimates is correlated with the oil price, which is highly likely as new exploration and extraction efforts are made and technologies developed in response to high oil prices, more efforts should be undertaken to hedge this risk in the above-ground portfolio. In practise, this will mean even less investment in the oil and gas industry is optimal. Evidently, the social benefit of a windfall should be calculated using a price corrected for the social cost of carbon.

Furthermore, the quantity of resources that should be extracted taking climate change into account requires careful consideration. World leaders have agreed at the Paris International COP21 Conference on Climate Change to limit global

[12]Their model is based on the long-run risk model of (Bansal and Yaron 2004), also includes Poisson shocks to model disasters and has to be solved numerically. As in Chapter 5, their model incorporates "stochastic differential utility" or "recursive preferences" (Duffie and Epstein 1992), so that the decision maker displays a preference for early resolution of risk (if $CRRA > 1/EIS$).

[13]There is a large literature on optimal extraction under uncertainty, with seminal contributions by Pindyck (1978) and Pindyck (1980), which I do not review here. Recently, Anderson et al. (2014) have proposed an alternative to the classical Hotelling model (Hotelling 1931), in which exploration is chosen optimally and production is driven by geological constraints and have used it to explain observed price patterns.

warming to 2°C with a goal of eventually lowering this further to 1.5°C above pre-industrial temperatures. McGlade and Ekins (2006) have shown that one third of all oil reserves need to be kept unexploited and half of gas reserves kept in the ground to meet the 2°C target.[14]

The risk of stranded assets does not just arise because of climate change. In the Dutch province of Groningen production of natural gas, which has historically been an important source of revenue for the national government (Wierts and Schotten 2008), has recently been reduced in response to prevalent earthquakes resulting from extraction in Groningen. As natural revenues only constitute a small share of GDP in the Netherlands, the precautionary motive is likely small. Had the risk of earthquakes been known in the initial years of extraction and judged significant, the precautionary motive during the years of high revenues (1980s) would have been larger, perhaps providing incentive for the creation of a Norwegian-style sovereign wealth fund, which never was.

6.3 Climate uncertainty and catastrophes

Chapter 5 has considered two main sources of uncertainty specific to climate change: the extent of warming (for a given increase in the carbon stock) and the damages to the economy (given an increase in temperature).[15] In Chapter 5 it was assumed that all uncertain variables are described by normally distributed processes (Brownian motions). Nonlinear (power-law) transformations of these processes led to changes in the first three statistical moments (mean, variance and skewness), but left the high-temperature tail fundamentally unchanged. Furthermore, solutions were sought in terms of power-series expansions with the fraction of climate damages in GDP as the small parameter. What are the implications of these assumptions?

Of these two types of uncertainty, uncertainty in the extent of climate damages is least well understood. The damage specifications used most commonly in integrated assessment models[16] are ad hoc and little more than extrapolations when used for high temperatures.[17] The challenge, to capture the response of all the moving parts of the world economy in response to future temperature changes in the form of a simple parametric dependency of world GDP on average world temperatures never observed, is indeed phenomenal. Two different approaches have been taken in the literature to estimate damage functions. Dell et al. (2012),

[14]For example, Rezai and van der Ploeg (2017) have examined how fast and how much the transition to carbon-free energy needs to occur.

[15]The role of a third degree category of uncertainty, volatility of the carbon stock, was found to be small.

[16]E.g. Nordhaus (2008), Ackerman and Stanton (2012) or DICE2013R (see Appendix D.6.5).

[17]See also the discussion in Pindyck (2013b).

Bansal and Ochoa (2011b) and Bansal and Ochoa (2011a) have used historical fluctuations in temperature within countries to identify their effects on aggregate economic outcomes (consumption and asset prices, as well as welfare) based on approximately half a century of data and fluctuations mostly smaller than 1°C. Very recently and taking an alternative approach, Hsiang et al. (2017) have used a broad inventory of damage estimates in different sectors and from across different fields of research, to estimate effects on aggregate economic outcomes.[18]

Hsiang et al. (2017) have found that the combined value of market and non-market damages increases quadratically in global mean temperature, costing roughly 1.2% of GDP per +1°C on average. Specifically, they estimate the relationship $D = 0.283\Delta T + 0.146\Delta T^2$ for long-term damages (in 2080-2099), where ΔT is the temperature in 2080-2099 relative to the temperature in 1980-2010 (in °C). I set $\Delta T = T - 0.65$ with T denoting the temperature relative to pre-industrial. Figure 6.1 shows this damage specification and the two damage specifications based on a power-law function proposed in Chapter 5 (continuous black lines), corresponding to proportional damages (Figure 6.1a: $D \propto T^2$, $D \propto E$) and convex damages (Figure 6.1b: $D \propto T^{2.5}$, $D \propto E^{1.25}$).[19], as well as commonly used damage specifications (see Appendix D.6.5 for further details). It is evident from this figure that the estimates by Hsiang et al. (2017) lie below what I have used in Chapter 5 and below other commonly used specifications, especially for high temperatures.[20]

Crucially, in Chapter 5, uncertainty in climate damages does not change the risk-adjusted social cost of carbon, unless it is correlated with the rest of the economy. To explain why, note that the estimates in Chapter 5 are of leading-order in the small parameter ϵ, which denotes the ratio of climate damages to GDP. Terms that result from the joint effect of a change in marginal utility due to climate damages and the climate damages themselves would be proportional to their product, second order in ϵ and thence ignored at leading order. It is instructive to consider the present value of marginal climate damages, expressed using the stochastic discount factor:

$$P(t) \propto E_t\left[\int_t^{\infty} e^{-\rho\tau}\frac{U'\left(C^{(0)}(\tau)+\epsilon C^{(1)}(\tau)+...\right)}{U'(C(t))}\underbrace{\frac{\partial}{\partial E}}_{O(\epsilon)}\left(Y^{(0)}(\tau)+...\right)d\tau\right], \quad (6.3.1)$$

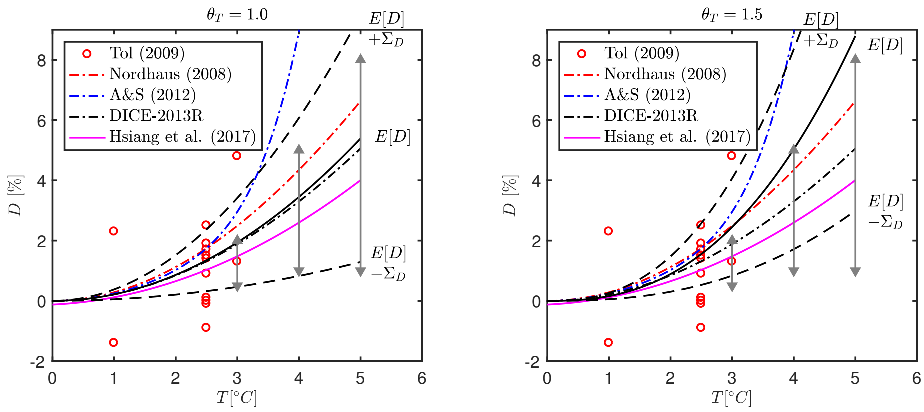
[18] Hsiang et al. (2017) have developed what they call their Spatial Empirical Adaptive Global-to-Local Assessment System (SEAGLAS): a "flexible architecture for computing damages that integrates climate science, econometric analyses, and process models." They combine market and non-market damage estimates across sectors, including agriculture, crime, coastal storms, energy, human mortality, and labour.

[19] Underlying this is $T \propto E^{0.5}$, and the terms 'proportional' and 'convex' refer to the dependence on the atmospheric carbon stock E, measured relative to pre-industrial.

[20] The confidence bands I have obtained in Chapter 5 based on the survey of estimates in Tol (2009) are also much wider than the confidence bands in Figure 5a of Hsiang et al. (2017).

Figure 6.1: Climate damage calibration: comparison with new results by Hsiang et al. (2017).

(a) Proportional damages (b) Convex damages



where consumption C in the stochastic discount factor is explicitly expanded as a series solution in ϵ. Since marginal climate damages $\partial Y/\partial E$ are already first order in ϵ, the terms in the stochastic discount factor can be taken to include only zeroth-order terms and thus exclude the effect of climate change when leading-order solutions are sought. The corrections in the risk-adjusted carbon price that capture the decision maker's risk aversion and prudence thus only arise because of the volatility of the non-climatic part of the economy itself or its correlation with the climatic uncertainties, but never from the climatic uncertainties alone. In non-technical terms, the justification for ignoring these terms is that they are simply too small, which is supported by standard damage specifications in which damages are a small share of GDP (cf. Figure 6.1). This share is typically only a few percent and always smaller than 10%. Its square would be very small indeed and too small to merit attention (cf. $0.1^2 \sim 0.01$). There are two important caveats to this assumption.

First, Dell et al. (2012), Bansal and Ochoa (2011b) and Bansal and Ochoa (2011a) have found temperature affects the growth rate and not the level of GDP, as considered in most integrated assessment models (and the model in Chapter 5). As a result, Bansal and Ochoa (2011b) and Bansal et al. (2016) find that temperature shocks do carry a positive risk premium when agents have a preference for early resolution of uncertainty (if $CRRA > 1/EIS$) and that this risk premium can be significant.

Second, an asymptotic analysis with the ratio of climate damages to GDP as the small parameter automatically ignores the effect of climate catastrophes.

Physically, climate catastrophes could take the form of non-marginal productivity shocks, such as flooding of cities and the sudden increased occurrence of storms and droughts (Alley et al. 2003), or climate tipping points, when the climate system shifts to a different equilibrium, such as abrupt desertification of agricultural land; reversal of the Gulf Stream; irreversible melt of the Greenland ice sheet; or dieback of the Amazon rainforest (Lenton 2011). For these low-probability, high-impact effects to be accounted for in integrated climate assessment, the tail of the probability must be carefully considered.[21]

Weitzman (2009, 2011) has shown that a sufficiently fat tail of the probability distribution of climate outcomes necessitates very stringent carbon policy. Indeed, even for a normally distributed process that is amplified by a linear feedback, the tail can become so fat that no statistical moments are defined (e.g. Roe and Baker (2007)). Any assessment that is based on evaluating the expected outcome after transformation by a standard utility function will be ill defined, as moments diverge. In popular terms, we must avoid climate change at all cost, for else we will all die. We are no longer in a regime where costs can be weighed against benefits. Other social and environmental problems no longer matter. Instead, the climate will pose a constraint of 1.5 or 2°C warming, and optimal policy should avoid this cliff-edge or point-of-no-return by minimization the costs of intervention (e.g. Aengenheyster et al. (2018).

Although there is some consensus that equilibrium probability distributions are positively skewed (IPCC 2013), it is not all clear exactly how much of the probability mass is located at very high temperatures. Although there is data on previous non-climatic catastrophic events (Barro 2006, Pindyck and Wang 2013), such data is simply not available for the climate. It is important to note that some of the fat-tailed distributions (e.g. Roe and Baker (2007)) do not have much probability mass at large, yet realistic temperatures. The unboundedness of their moments relies on an algebraically decaying tail that gives diverging contributions only when integrated to inconceivably large temperatures. Indeed, Pindyck (2011) has shown that Weitzman's (2009) dismal theorem can be resolved by bounding marginal utility. How to deal with the question of tails is one of (informed) judgement, and the solution proposed in this thesis – based on normally distributed processes that are transformed through power-law functions to display skewness, but with the effects of kurtosis and higher-order moments curtailed by a perturbation expansion – relies on a perhaps intuitive combination of the central limit theorem and Occam's razor.

[21] An increasing number of authors have considered the effect of climate tipping points in the context of the DICE integrated assessment model (Nordhaus 2008) (e.g. Keller et al. (2004), Lemoine and Traeger (2014), Cai et al. (2015), Lontzek et al. (2015)). Recently, van der Ploeg and de Zeeuw (2018) have done so in the context of a growth model and have considered the role of precautionary capital build-up.

6.4 Conclusions

Concluding, this thesis has shown what might be components of an optimal, that is prudent and precautionary, response to natural resource and climate uncertainty in the framework of expected utility. Future work may readily extend the closed-form expressions from asymptotic theory for the optimal risk-adjusted social cost of carbon put forward in Chapter 5 to include long-term risk, Poisson shocks to model disasters and higher-order terms in the asymptotic expansion to capture the effect on welfare of these low-probability, high-impact events. The resource extraction and financial portfolio allocation decisions in Chapter 4 should be integrated into the climate model of Chapter 5 to give four instruments to simultaneously manage a natural resource windfall and the detrimental effects of climate change: the resource extraction rate, the (precautionary) savings rate, the portfolio allocation and the carbon tax. Such a model should allow for a full, time-varying covariance structure between resource prices, asset prices and economic growth, climate and climate damage uncertainty.

One important factor has not been taken into account: inequality. This thesis has considered intertemporal inequality aversion, but not intratemporal or social inequality aversion. The persistence of high poverty rates in many resource-rich economies makes the question of how windfalls affect the distribution of income between the rich and the poor as relevant as the distribution between generations considered in this thesis. In fact, Goderis and Malone (2011) have found that uncertainty about future commodity prices increases long-term inequality. For many countries, natural resources may be a curse (van der Ploeg and Poelhekke 2009).

In the case of climate change, put simply, the rich pollute, and the poor suffer. This inequality is generally not captured by the climate damage function.[22] For the United States, Hsiang et al. (2017) have obtained spatially dependent estimates of economic damage from climate change. Falling agricultural yields and labour productivity and rising mortality and crime are found to be large in the hot southern counties, where incomes are below the national average. In richer New England and the Pacific north-west, in contrast, winters will be milder and less deadly, and agricultural yields may rise. By the late 21st century, under business-as-usual emissions the average very likely range (90% confidence interval) for damages is 1.2 to 6.8% of county income in the richest third of counties, whereas for the poorest third of counties the average range is 2.0 to 19.6% of county income (Hsiang et al. 2017).[23] Without extending these results to the differences between different countries across the developed and developing world, estimates of the aggregate economic cost of climate change will be too low and the social cost underestimated.

[22] There is a literature addressing whether who should pay for climate change abatement, e.g. Chichilnisky and Heal (1994) and D'Autume et al. (2016).

[23] These differences are more extreme for the richest 5% and poorest 5% of counties, with average intervals for damage of 1.1 to 4.2% and 5.5 to 27.8%, respectively.

Summary

This thesis examines the optimal macroeconomic policy response to the uncertainty associated with natural resource revenues and the uncertainty associated with climate change. Both areas lie outside the realm of day-to-day household behaviour and require explicit government intervention, which is subject to fundamental normative assumptions affecting the valuation of costs and benefits occurring in an uncertain future. If a decision maker is prudent, that is, less risk averse at higher income levels, uncertainty about future income leads to additional saving and postponing of consumption.

For natural resource uncertainty, prudent policy takes the form of additional (precautionary) saving of resource revenues, often in a sovereign wealth fund. The investments in such a fund must be optimally allocated in the face of uncertainty in the financial markets, taking heed of their correlation with the value of underground assets. For climate uncertainty, prudent policy generally leads to a higher optimal carbon tax, reflecting a higher social cost of emitting carbon. Yet, the effects of economic and climatic uncertainty are distinct and require a careful consideration of the risk aversion implicit in the climate model and the climate damage function.

Starting from a continuous-time, time-separable, dynamic stochastic welfare optimisation framework, this thesis uses perturbation methods to develop leading-order estimates of the effect of these two types of uncertainty in combination with risk aversion and prudence. Its objectives are two-fold: improve our understanding of the mechanisms through which uncertainty acts and provide order-of-magnitude estimates of the effects and thus be able to assess their importance within the broader context of macroeconomic policy.

Chapter 2 Precautionary saving for resource price uncertainty

Chapter 2 examines the role of precautionary saving in the optimal management of natural resource windfalls. It addresses the question how resource-rich countries should allocate the temporary and highly volatile income they receive from extracting natural resources. This income, sometimes referred to as windfall income because of its temporary nature, can be either consumed, invested or saved. Generally, three types of funds may necessary to manage an oil windfall: intergenerational, liquidity, and investment funds. Precautionary saving for resource price uncertainty drives the size of the liquidity fund, and the optimal liquidity fund is larger if the windfall lasts longer and oil price volatility, prudence, and the GDP share of oil rents are high and productivity growth is low. The theoretical insights obtained in this chapter are applied to the windfalls of Norway, Iraq, and Ghana. The optimal size of Ghana's liquidity fund is tiny even with high prudence. Norway's liquidity fund is larger than Ghana's. Iraq's liquidity fund is colossal relative to its intergenerational fund. Only with capital scarcity, should part of the windfall be used for investing to invest. For developing economies such as Ghana, where public capital is scarce and debt burdens are high, paying off sovereign debt and investing in public capital may be more important driving forces than precautionary saving.

Chapter 3 Case study: resource revenues in Alberta

Through a case study of the Canadian province Alberta, Chapter 3 examines the policy implications of an uncertain natural resource windfall for government finances in particular. Based on an equivalent welfare-based intertemporal stochastic optimization model to Chapter 2, Chapter 3 estimates the size of the optimal intergenerational and liquidity funds and the corresponding resource dividend available to the Albertan government. To leading order, this dividend should be a constant fraction of total above- and below-ground wealth, complemented by additional precautionary savings at initial times to build up a small liquidity fund to cope with oil price volatility. Finally, the effect of the 2014 plunge in oil prices on our estimates is examined.

Chapter 4 Asset allocation and extraction for resource SWFs

Chapter 4 considers the important source of uncertainty ignored in Chapters 2 and 3: the asset return uncertainty of the sovereign wealth fund in which the proceeds from natural resource extraction are invested. One of the most important developments in international finance and resource economics in the past twenty years is the rapid and widespread emergence of the $6 trillion sovereign wealth fund industry, many of which are derived from natural resource rents. Oil exporters typically ignore below-ground assets when allocating these funds, and ignore above-ground assets when extracting oil. This chapter presents a unified stylized framework for considering both. Subsoil oil should alter a fund's portfolio through additional leverage and hedging. First-best spending should be a share of total wealth, and any unhedgeable volatility must be managed by precautionary savings. An optimally chosen financial portfolio will reduce the aggregate level of uncertainty to which the economy is exposed, by choosing assets that offset oil price risk. If such a portfolio is unavailable, additional precautionary saving may be required. If oil prices are pro-cyclical, oil should be extracted faster than the Hotelling rule to generate a risk premium on oil wealth. Finally, this chapter discusses how its analysis could improve the management of Norway's fund in practice.

Chapter 5 The risk-adjusted carbon price

Moving onto climate uncertainty, Chapter 5 examines the effect of uncertainty on estimates of the social cost of carbon and thence the optimal carbon tax. The existing and popular model of the economy and climate change by Golosov et al. (2014) has logarithmic preferences and damages proportional to the carbon stock in which case the certainty-equivalent carbon price is optimal. This chapter allows for different aversions to risk and intertemporal fluctuations, convex damages, uncertainties in economic growth, atmospheric carbon, climate sensitivity and damages, correlated risks, and distributions that are skewed in the longer run to capture climate feedbacks. This chapter thus derives a non-certainty-equivalent rule for the carbon price, which incorporates precautionary, risk-insurance and risk-exposure, and climate-beta effects to deal with future economic and climatic risks. This is achieved in the context of a stylized integrated assessment model based on an endogenous growth model. A combination of different perturbation methods is used to develop simplified rules for the social cost of carbon and its dependence on four categories of uncertainty: shocks to the carbon cycle, uncertain climate sensitivity and damage function estimates and, finally, the uncertain evolution of total factor productivity. Quantitative estimates of the risk-adjusted carbon price are obtained after calibration of the model.

Samenvatting

Dit proefschrift onderzoekt de optimale macro-economische beleidsaanpak van de onzekerheid van inkomsten uit natuurlijke hulpbronnen en de onzekerheid van klimaatverandering. Beide bevinden zich buiten het domein van het dagelijks handelen van huishoudens en vereisen expliciet overheidsingrijpen, hetgeen onderhavig is aan fundamentele normatieve aannames met invloed op de waardering van kosten en baten in een onzekere toekomst. Als de beleidsmaker prudent is, dat wil zeggen minder risico-avers bij een hoger inkomensniveau, dan leidt onzekerheid over toekomstig inkomen tot extra sparen en het uitstellen van consumptie.

In het geval van onzekerheid omtrent natuurlijke hulpbronnen neemt prudent beleid de vorm van extra (voorzorgs) sparen van de inkomsten uit dergelijke hulpbronnen, vaak in een staatsvermogensfonds. De beleggingen in een dergelijk fonds moeten optimaal worden toegewezen in de context van de onzekerheid op financiële markten, rekening houdend met hun correlatie met de waarde van ondergrondse bezittingen. Voor klimaatonzekerheid leidt een prudent beleid doorgaans tot een hogere optimale CO_2-belasting, wat een weerspiegeling is van hogere maatschappelijke kosten van het uitstoten van CO_2. De effecten van economische en klimatologische onzekerheid zijn echter verschillend en vereisen een zorgvuldige afweging van de risico-aversie die impliciet is in het klimaatmodel en in de specificatie van klimaatschade.

Dit proefschrift gebruikt een dynamisch stochastisch kader in continue en scheidbare tijd voor optimalisatie van welvaart en maakt gebruik van perturbatietheorie om schattingen te maken van de orde van grootte van het effect van deze twee typen onzekerheid in combinatie met risico-aversie en prudentie. De doelstellingen zijn tweevoudig: verbeteren van ons begrip van de mechanismen waarmee onzekerheid haar werk doet en het maken van schattingen van de orde van grootte van effecten om derhalve het belang ervan te kunnen beoordelen binnen de bredere context van macro-economisch beleid.

Hoofdstuk 2 Voorzorgssparen voor prijsonzekerheid van natuurlijke hulpbronnen

Hoofdstuk 2 onderzoekt de rol van voorzorgssparen bij het optimale beheer van tijdelijke inkomsten uit natuurlijke hulpbronnen. Het gaat over de vraag hoe grondstofrijke landen het tijdelijke en zeer volatiele inkomen dat zij ontvangen uit de winning van natuurlijke hulpbronnen moeten besteden. Dit vaak tijdelijke inkomen kan worden geconsumeerd, geïnvesteerd of gespaard. Over het algemeen zijn er derhalve drie soorten fondsen nodig: intergenerationele, liquiditeits- en investeringsfondsen. Voorzorgssparen voor onzekerheid over de prijs van grondstoffen drijft de omvang van het liquiditeitsfonds op, en het optimale liquiditeitsfonds is groter als de inkomsten van langere duur zijn en de olieprijsvolatiliteit, prudentie en het bbp-aandeel van de olie-inkomsten hoog zijn en de productiviteitsgroei laag. De theoretische inzichten verkregen in dit hoofdstuk worden toegepast op de olie- en gas-inkomsten van Noorwegen, Irak en Ghana. De optimale omvang van het liquiditeitsfonds van Ghana is klein, zelfs met een hoge mate van prudentie. Het liquiditeitsfonds van Noorwegen is groter dan dat van Ghana. Irak's liquiditeitsfonds is kolossaal vergeleken met zijn intergenerationeel fonds. Alleen met kapitaalschaarste moet een deel van de olie-inkomsten worden gebruikt om te investeren. Voor ontwikkelende economiën zoals Ghana, waar publiek kapitaal schaars is en de schuldenlast hoog is, kunnen het betalen van staatsschulden en het investeren in publiek kapitaal een belangrijker drijfveer zijn dan voorzorgssparen.

Hoofdstuk 3 Casestudy: inkomsten uit natuurlijke hulpbronnen in Alberta

Door middel van een casestudy van de Canadese provincie Alberta onderzoekt Hoofdstuk 3 de beleidsimplicaties van tijdelijke onzekere inkomsten uit natuurlijke hulpbronnen voor de overheidsfinanciën in het bijzonder. Gebaseerd op hetzelfde, op welvaart gebaseerde intertemporeel stochastisch optimalisatiemodel als in Hoofdstuk 2, raamt Hoofdstuk 3 de omvang van de optimale intergenerationele en liquiditeitsfondsen en het bijbehorende grondstoffendividend voor de overheid van Alberta. Dit dividend moet een ongeveer constant deel uitmaken van het totale vermogen boven en onder de grond, aangevuld met extra voorzorgssparen in het begin om een klein liquiditeitsfonds op te bouwen om de schommelingen van de olieprijzen aan te kunnen. Tenslotte wordt het effect van de daling van de olieprijzen in 2014 op de schattingen onderzocht.

Hoofdstuk 4 Asset-allocatie en grondstoffenwinning voor staatsvermogensfondsen met natuurlijke hulpbronnen

Hoofdstuk 4 beschouwt de belangrijke bron van onzekerheid genegeerd in Hoofdstukken 2 en 3: de onzekerheid van het rendement van het staatsvermogensfonds waarin de opbrengsten van de winning van natuurlijke rijkdommen zijn belegd. Een van de belangrijkste ontwikkelingen in de internationale financiële en grondstoffeneconomie in de afgelopen twintig jaar is de snelle en wijdverspreide opkomst van staatsvermogensfonden (sovereign wealth funds) met een gezamenlijke waarde van \$6 biljoen, waarvan vele hun inkomsten werven uit natuurlijke hulpbronnen. Olie-exporteurs negeren doorgaans ondergrondse assets bij de toewijzing van beleggingen in deze fondsen en negeren bovengrondse assets bij het winnen van olie. Dit hoofdstuk presenteert een uniform gestileerd raamwerk voor beide. Olie in de grond zou de portefeuille van een fonds moeten veranderen door extra hefboomwerking en hedging. De optimale bestedingen moeten een aandeel zijn in de totale welvaart, en elke niet-verhandelbare volatiliteit moet worden gemitigeerd door voorzorgssparen. Een optimaal gekozen financiële portefeuille zal de geaggregeerde onzekerheid waaraan de economie blootstaat verminderen door assets te kiezen die het olieprijsrisico compenseren. Als een dergelijk portfolio niet beschikbaar is, kan extra voorzorgssparen vereist zijn. Als de olieprijzen procyclisch zijn, moet olie sneller worden gewonnen dan de Hotelling-regel om een risicopremie te genereren voor de oliewaarde. Tenslotte wordt in dit hoofdstuk besproken hoe de analyse ervan het beheer van het Noorse staatsvermogensfonds in de praktijk kan verbeteren.

Hoofdstuk 5 De voor risico gecorrigeerde CO_2 prijs

Hoofdstuk 5 onderzoekt het effect van onzekerheid op schattingen van de maatschappelijke kosten van CO_2 en de overeenkomstige optimale CO_2-belasting. Het bestaande en populaire model van de economie en klimaatverandering door Golosov et al. (2014) heeft een logaritmische nutsfunctie en een klimaatschadespecificatie evenredig aan de hoeveelheid CO_2 in de atmosfeer, in welk geval de zekerheids-equivalente CO_2-prijs optimaal is. Dit hoofdstuk generaliseert dit resultaat voor verschillende aversies tegen risico's en intertemporele fluctuaties, convexe klimaatschade, onzekerheden in economische groei, atmosferische CO_2, klimaatsensitiviteit en klimaatschade, gecorreleerde risico's en kansverdelingen die op de langere termijn scheef worden door klimaatterugkoppeling. Dit hoofdstuk leidt een niet zekerheids-equivalente regel af voor de CO_2-prijs, die prudentie, risicoverzekering en risicoblootstelling omvat, en klimaat-bèta-effecten om toekomstige economische en klimatologische risico's het hoofd te bieden. Dit wordt bereikt in de context van een gestileerd geïntegreerd beoordelingsmodel (integrated assessment model) op basis van een endogeen groeimodel. Een combinatie van verschillende perturbatiemethoden wordt gebruikt om vereenvoudigde regels te ontwikkelen voor de maatschappelijke kosten van CO_2 en de afhankelijkheid van vier categorieën van onzekerheid: schokken in de CO_2-cyclus, onzekere klimaatsensitiviteit en schattingen van de klimaatschadespecificatie en, tenslotte, de onzekere evolutie van de totale factor productiviteit. Kwantitatieve schattingen van de voor risico gecorrigeerde CO_2-prijs worden verkregen na kalibratie van het model.

Biography

Ton Stefan van den Bremer was born in Groningen on 28th May 1987, where he attended the Groningse School Vereniging (GSV) and the Praedinius Gymnasium. He obtained an MEng in Civil Engineering (first class honours) from Imperial College London (2005-2009), winning the Unwin Medal and Prize, the Institution of Civil Engineers Prize, the Sir Bruce White Prize, the Letitia Chitty Prize and the Student Centenary Prize. Afterwards, Ton obtained an MPhil in Economics from the University of Oxford, studying at St Antony's College Oxford.

From 2012 to 2014, Ton completed a DPhil in Engineering Science from the University of Oxford, studying at New College. His thesis, supervised by Prof. P.H. Taylor, examined how the periodic motion of different types of gravity waves can be responsible for the net transport of tracers such as particles, momentum and energy with applications for pollutant dispersion in the ocean and internal wave stability in the atmosphere. During this time, Ton was a College Lecturer at Keble College, University of Oxford, a Geophysical Fluid Dynamics Fellow at the Woods Hole Oceanographic Institution (U.S.) and was awarded an ERCOFTAC Osborne Reynolds Student Award (2015) for his DPhil research.

From January to September 2015, Ton worked as a Researcher in the School of Business and Economics at the Vrije Universiteit Amsterdam, Adjunct Lecturer in Environmental Economics at University College Amsterdam and Promovendus under the supervision of Professor F. van der Ploeg. In October 2016, Ton became a Chancellor's Fellow in the School of Engineering at the University of Edinburgh, where he taught Fluid Mechanics and Computational Methods. During this time, he was a Visiting Professor in the Department of Physics at the University of Alberta and was awarded a Royal Academy of Engineering Research Fellowship.

Since January 2018, Ton has been an Associate Professor in Fluid Mechanics and a Royal Academy of Engineering Research Fellow in the Department of Engineering Science and the Edward and Catherine Wray Tutor and Fellow at Worcester College, University of Oxford. He is also a Visiting Professor at the University of Edinburgh and lives in Oxford and Amsterdam.

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Appendix A

Appendix to Chapter 3: Model equations

Empirical evidence (e.g. Hamilton (2009)) suggests that it is hard to reject the hypothesis that the crude oil price follows a random walk (a Brownian Motion process in continuous time). However, it is also not possible to reject a high degree of persistence with mean reversion (van den Bremer and van der Ploeg 2013). To avoid heteroskedasticity of the standard errors, we thus assume an AR(1) stochastic processes for the logarithm of the price of bitumen, conventional oil or natural gas (Schwartz 1997):

$$dP_i(t) = (\eta_i [\mu_i + \upsilon_i t - \log (P_i(t))] + \upsilon_i) P_i(t)dt + \sigma_i P_i(t)W(t), \tag{A.0.1}$$

where μ_i is the mean, σ_i the volatility, υ_i the drift, η_i the rate of mean reversion and $W(t)$ a Wiener process. Equation (A.0.1) can be written as an AR(1) stochastic process:

$$d \log (P_i(t)) = \left(\eta_i \left[\mu_i^* + \upsilon_i t - \log (P_i(t))\right] + \upsilon_i\right) dt + \sigma_i dW(t), \tag{A.0.2}$$

where $\mu_i^* \equiv \mu_i - 0.5\sigma_i^2/\eta_i$. Using three correlated stochastic processes for bitumen, conventional oil and natural gas, we can write the volatility of the dividend as:

$$\begin{aligned} D^2\sigma_D^2 = {} & \left(\frac{\partial D}{\partial P_B}\right)^2 P_B^2\sigma_B^2 + \left(\frac{\partial D}{\partial P_O}\right)^2 P_O^2\sigma_O^2 + \left(\frac{\partial D}{\partial P_G}\right)^2 P_G^2\sigma_G^2 \\ & + 2\frac{\partial D}{\partial P_B}\frac{\partial D}{\partial P_O} P_B P_O \rho_{BO}\sigma_B\sigma_O + 2\frac{\partial D}{\partial P_B}\frac{\partial D}{\partial P_G} P_B P_G \rho_{BG}\sigma_B\sigma_G \\ & + 2\frac{\partial D}{\partial P_O}\frac{\partial D}{\partial P_G} P_O P_G \rho_{OG}\sigma_O\sigma_G, \end{aligned} \tag{A.0.3}$$

where σ_B, σ_O and σ_G are the volatilities of the prices of bitumen, conventional oil and natural gas and ρ_{BO}, ρ_{BG} and ρ_{OG} are the correlations between the respective price processes.

Instead of solving the system of partial differential equations numerically, we use the solutions to the deterministic solution to solve the problem approximately and obtain insight into the role of uncertainty. Formally, this approach would correspond to taking the first-order term in a Taylor-series expansion with the volatility σ_i as the small parameter, where the zeroth-order term would correspond to the deterministic solution (see also van den Bremer and van der Ploeg (2013)). We thus have from (3.2.4) that the effect of a shock at time t on the resource dividend at that same time is the net present value of all future effects of this shock:

$$\frac{\partial D(t)}{\partial P_i(t)} = [r - \theta(r-\rho) - n] \int_t^{\infty} \frac{\partial E_t[P_i(\tau)]}{\partial P_i(t)} O_i(\tau) e^{-r(\tau-t)} d\tau, \qquad \text{(A.0.4)}$$

where the price process in (A.0.1) gives:

$$\frac{\partial E_t[P_i(\tau)]}{\partial P_i(t)} = \begin{cases} 1 & \text{random walk,} \\ \dfrac{\exp\left(\mu_i\left[1 - e^{-\eta_i(\tau-t)}\right] + \log(P_i(t))e^{-\eta_i(\tau-t)}\right)}{P_i(t)} e^{-\eta_i(\tau-t)} & \text{AR(1).} \end{cases} \qquad \text{(A.0.5)}$$

It is evident then from (A.0.4) and (A.0.5) that mean reversion acts to decrease the effect of a price shock of the corresponding increment in the resource dividend and thus reduces the need for precautionary savings. These equations are analogous to those in Chapter 2.

Appendix B

Appendix to Chapter 3: Detailed description of data

B.1 Real interest rates

Over the 2002-2012 period, the average real annual rate of return on the Alberta Heritage Savings Trust Fund was 6.1 per cent per year with a nominal rate of return of 8.1 per cent per year [1] and average inflation in that period of 2.0 per cent per year,[2] compared to 3.7 per cent per year for the Norwegian Pension Fund Global over the same period.[3]

The average real annual interest rates on Canadian and U.S. government bonds with maturities of 1 year, 5 years and 10 years over the same period were 0.4 per cent per year for Canadian and −0.6 per cent for U.S.; 1.1 per cent per year for Canadian and 0.4 per cent per year for U.S.; and 1.7 per cent per year for Canadian and 1.2 per cent per yearfor U.S.[4] For the 1992-2012 period, the same rates were 1.9 per cent for Canadian (0.8 per cent U.S.), 2.7 per cent (1.7 per cent) and 3.2 per cent (2.7 per cent). The average real annual interest rates paid on Alberta provincial debt over the period 2005-2012 were 0.1 per cent, 1.3 per cent per year and 2.2 per cent per year on bonds with maturities of 1 year, 5 years and 10 years respectively.[5] We set r = 6.1 per cent per year and abstract from the risky nature of the returns.

[1] Alberta Treasury Board and Finance, "Heritage Fund Annual Report 2012-2013, www.finance.alberta.ca/business/ahstf/publications.html.

[2] Statistics Canada, Cansim Online Statistics Database (2013), http://www5.statcan.gc.ca/cansim/home-accueil?lang=eng&p2=49&MM.

[3] Norges Bank Investment Management, Annual Report 2012, http://www.nbim.no/Global/Reports/2012/Annual%20report/Annual%20report%202012.pdf.

[4] Bank of Canada, Private Communication (2013); Statistics Canada, Cansim Online Statistics Database (2013); U.S. Federal Reserve, "Historical data on selected interest rates," (Washington, D.C.: 2013), http://www.federalreserve.gov/releases/h15/data.htm.

[5] Alberta Treasury Board and Finance, Private Communication (2013).

B.2 Estimates of reserve stocks

We use "remaining established reserves" as defined by the Alberta Energy Regulator ("recoverable quantities known to be left").[6] Remaining established reserves correspond approximately to proven reserves. We then allow for discoveries based on historical data. At the end of 2012, remaining established reserves are:[7]

- Bitumen (or oil sands): 168 billion barrels.
- Conventional oil (light and heavy crude): 1.7 billion barrels.
- Natural gas: 5.8 billion barrels of oil equivalent (916 billion SM^3).

We obtain the following R/P (reserves to production ratios) for 2012:

- *Bitumen*: at 2012 production rates of 0.72 billion bbl/year, we obtain a R/P ratio of 230 years. We do not allow for future discoveries.
- *Conventional oil* (encompassing light, heavy, and crude oil): at 2012 production rates of 0.20 billion bbl/year we obtain an R/P ratio of 8.5 years. Although for conventional oil there are significant new discoveries over many decades, production has only marginally exceeded new discoveries in the last 10 years with 10-year (20-year) averages of 0.19 (0.25) and 0.20 (0.20) billion bbl/year, respectively. To reflect this, we assume discoveries decline linearly from 0.20 billion bbl/year to zero in 30 years and increase the current reserves by 3.0 billion barrels accordingly.
- *Natural gas*: at 2012 production rates of 0.59 billion b.o.e./year, we obtain an R/P ratio of 10 years. We note significant new discoveries. Production has marginally exceeded new discoveries during the last 10 years with 10-year averages (20-year) of 0.78 (0.82) and 0.64 (0.60) billion b.o.e./year, respectively. We thus assume discoveries decline linearly from 0.64 billion b.o.e./year to zero in 30 years and increase current reserves by 9.6 billion b.o.e.

We exclude gas from oil wells (circa 10 per cent) and other natural resources such as coal and sulphur. Including our estimates for new discoveries, we use the following reserve estimates:

- Bitumen: 168 billion barrels.
- Conventional oil (light and heavy crude): 1.7 + 3.0 = 4.7 billion barrels.
- Natural gas: 5.8 + 9.6 = 15.4 billion b.o.e.

[6] Energy Resources Conservation Board (ERCB), ST98-2013 Alberta's Energy Reserves 2012 and Supply/Demand Outlook 2013-2022 (2013), `http://www.aer.ca/data-and-publications/statistical-reports`.

[7] ibid.

B.3 Official projections of extraction rates

Official projections are available until 2022.[8] In these official projections:

- *Bitumen* (or oil sands): production rates almost double and reach 1.4 billion bbl/year in 2021 from 0.72 billion bbl/year in 2012.
- *Conventional oil*: production rates decline marginally from 0.20 billion bbl/year in 2012 to 0.17 billion bbl/year in 2022.
- *Natural gas*: production rates decline from 0.58 billion b.o.e./year in 2012 to 0.44 billion b.o.e./year in 2022.

We use these official projections until 2022 and from then on we assume:

- *Bitumen* (or oil sands): in scenario 1 a continued linear increase of the production rate until 2.0 billion bbl/year in 2030 followed by a plateau at this rate of production until exhaustion in 2100; in scenario 2 production reaches a plateau in 2022 and continues at the constant rate of 1.44 bbl/year until exhaustion at a later time.
- *Conventional oil*: continued flat rate of production of 0.17 billion bbl/year until exhaustion in 2038.
- *Natural gas*: continued flat rate of production of 0.44 billion b.o.e./year until exhaustion in 2044.

B.4 Extraction costs

Van den Bremer and van der Ploeg[9] estimate that, apart from in the initial years 1970-75, when extraction costs were still very high as the very first exploratory and extraction activity took place, average extraction costs for Norway were U.S.$9/b.o.e. in the period 1990-2000, U.S.$6/b.o.e. for 2000-2005 and U.S.$14/b.o.e. for 2005-2010 (2013 prices). In the absence of data for extraction costs for conventional oil in Alberta, we thus set extraction costs to $15/bbl for conventional oil.

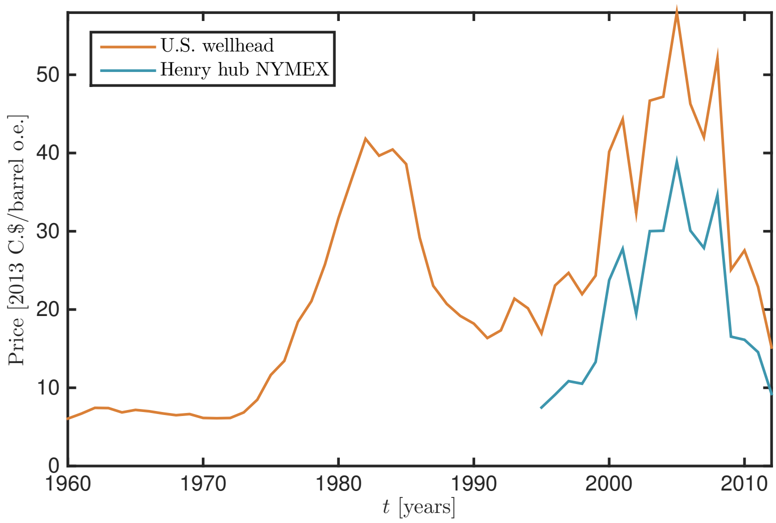
Extraction costs are significantly higher for bitumen. The Canadian Energy Research Institute[10] provides estimates of the extraction costs (calculated from subtracting its estimates of royalties and income taxes from its estimates of total supply costs; see its Figure E.1) for four different types of plants: 23, 36, 79 and 51 2011-WTI equivalent U.S.$/bbl. (i.e. the price at which extraction would just

[8] ibid.

[9] van den Bremer and van der Ploeg (2013).

[10] Canadian Energy Research Institute, "Canadian oil sands supply costs and development projects (2012-2046)" (2013), `http://www.ceri.ca/images/stories/2013-06-10_CERI_Study_133_-_Oil_Sands_Update_2012-2046.pdf`.

Figure B.1: Real natural gas prices (2013 prices).



become profitable ignoring taxes and royalties). The Canadian Energy Research Institute[11] assumes a constant price differential of 15 U.S.$/bbl between WTI and WCS, the price at which bitumen is sold following dilution for pipeline transportation. Ignoring diluent costs, we thus estimate extraction costs to be the average of these estimates minus the WTI-WCS price differential: 47 − 15 = \$32/bbl.

Furthermore, we note that there has been a significant increase in extraction costs in recent years. Comparing estimates of supply costs from 2012[12] of 72 U.S.$/bbl WTI equivalent averaging across different extraction methods to comparable estimates from 2005[13] of U.S.$35/bbl WTI equivalent reveals a twofold increase in costs in seven years (all prices are 2013 prices). To reflect this increase, we assume a linear increase from \$20/bbl in 2006 to \$32/bbl to calculate historical rents in Appendix B.7.

Due to the shale gas revolution in the U.S. and the resulting sharp decline in North American natural gas prices, many of the reserves in Western Canada are in fact not economical to extract at current natural gas prices. For natural gas, the Canadian Energy Research Institute[14] estimates extraction costs for vertical

[11] ibid.
[12] ERCB, "ST98-2013 Alberta's."
[13] ERCB, "ST98-2013 Alberta's."
[14] Canadian Energy Research Institute, "Conventional natural gas supply costs in western Canada" (2013), `http://www.ceri.ca/images/stories/ceri_study_136_-_conventional_natural_gas_supply_cost_-_final_june_2013.pdf`.

and horizontal extraction to be \$7.60/mcf and \$2.60/mcf or \$43 and \$20 /b.o.e.[15] The 2012 price of natural gas is below this at \$11/b.o.e. (see Figure B.1). Despite the large variation of extraction estimates across different extraction methods and across different Canadian provinces (\$2/mcf-\$10.20/mcf or \$11/b.o.e.-\$57/b.o.e.) provided by the Canadian Energy Research Institute,[16] we use \$15/b.o.e., corresponding to extraction costs of conventional oil.

We thus use the following extraction costs to calculate future resource rents:

- *Oil sands*: \$32/bbl.
- *Conventional oil* (light and heavy crude): \$15/b.o.e.
- *Natural gas*: \$15/b.o.e.

For natural gas, extraction costs may exceed prices, in which case we set resource rents to zero. Since we assume that Alberta gas is sold at the Henry Hub price and Alberta conventional oil is sold at the WTI price, we effectively abstract from transportation costs. Transportation costs only account for a small reduction in resource rents of the order of 5 per cent.

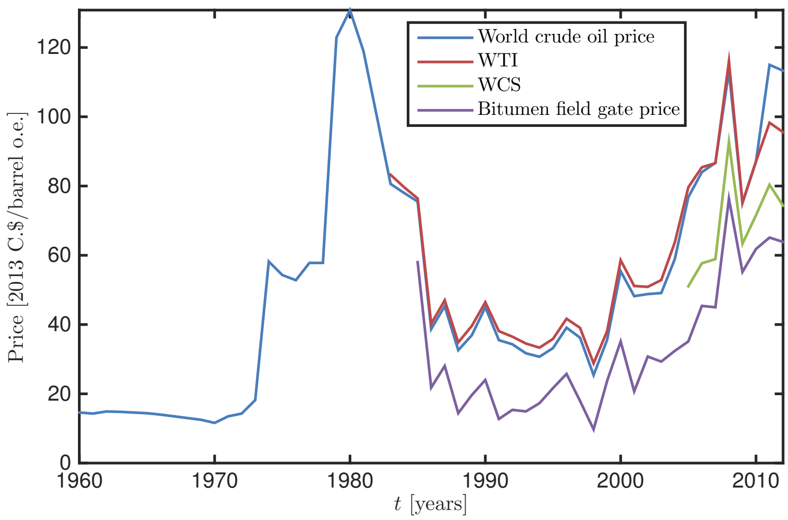
B.5 Price processes

Figure B.2 shows historical records of real oil and gas prices (discounted using Canadian CPI)[17] in Canadian dollars.[18]We assume conventional oil is sold at the WTI price and natural gas at the Henry Hub NYMEX natural gas price.[19] Also shown are the longer historical records: the world crude oil price[20] and the U.S. natural gas wellhead price.[21] To calculate the value of a barrel of bitumen, the costs of diluting heavy crude to make it transportable via pipelines have to be taken into account. Western Canadian Select therefore only provides an upper bound to the actual bitumen price. The average field gate price for bitumen[22] provides our estimate of the actual price of bitumen.

We use the values of the mean-reversion and volatility for the oil and gas price as estimated in van den Bremer and van der Ploeg (2013) (there is approximate parity of U.S. and Canadian dollars in 2013):

[15] ibid.
[16] ibid.
[17] Statistics Canada, Cansim Online Statistics Database (2013), `http://www5.statcan.gc.ca/cansim/home-accueil?lang=eng&p2=49&MM`.
[18] Exchange rates from Statistics Canada, Cansim Online Statistics Database (2013).
[19] Government of Alberta, Department of Energy, Private Communication (2013).
[20] BP, Statistical Review (London: BP, 2013), `http://www.bp.com/en/global/corporate/about-bp/statistical-review-of-world-energy-2013.html`.
[21] USEIA, "Independent Statistics and Analysis" (Washington, D.C.: U.S. Energy Information Administration, 2013), `www.eia.gov/dnav/ng/hist/n9190us3a.htm`.
[22] Government of Alberta, Department of Energy (2013).

Figure B.2: Real oil and bitumen prices (2013 prices).



- *Conventional oil*: a mean price of \$110/bbl, a mean-reversion coefficient of 6.0 per cent per year, and a volatility of 26 per cent.

- *Bitumen*: the same mean-reversion coefficient and volatility as conventional oil, but a substantially lower mean price of \$80/bbl,[23] which assumes a constant price differential of \$15/bbl between WTI and WCS.

- *Natural gas*: a mean price of \$32/b.o.e., a mean-reversion coefficient of 6.0 per cent per year, and a volatility of 20 per cent.

We set the correlation coefficients between the different prices to one. The time horizon of our analysis starts January 1, 2013. We use the 2012 prices as the initial prices:

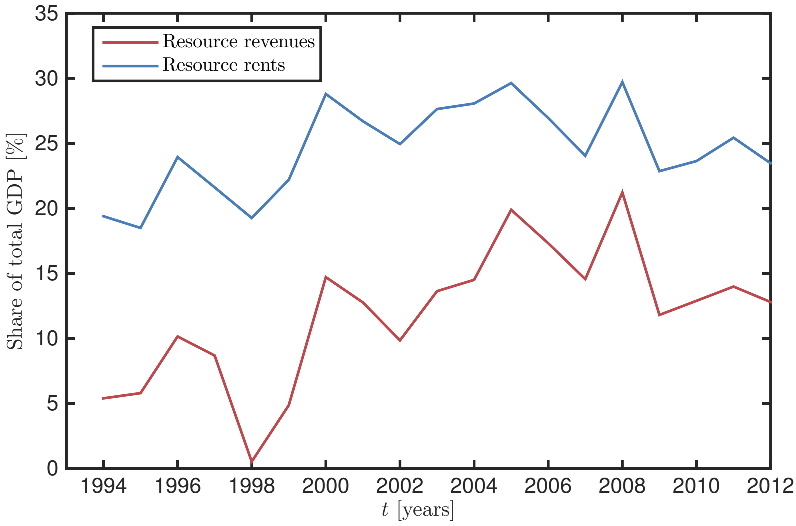
- *Conventional oil*: $P_O(t = 2013) = \$96$/bbl.

- *Bitumen*: $P_B(t = 2013) = \$64$/bbl.

- *Natural gas*: $P_G(t = 2013) = \$11$/b.o.e.

Extraction of natural gas will not be profitable in the initial years with extraction costs of 15\$/b.o.e., but will eventually become profitable as a result of reversion to the mean. All prices are in 2013 Canadian dollars unless otherwise indicated.

Figure B.3: Resource revenues and resource rents as a share of total GDP.



B.6 Economic and population growth

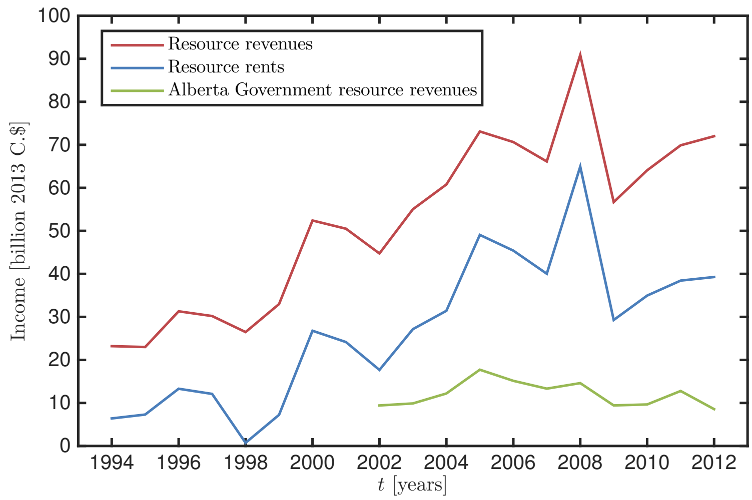
Alberta has seen relatively high rates of population growth, with a 10-year average of 2.2 per cent growth per year and 20-year average of 2.0 per cent growth per year,[24] but growth is forecast to decline to 1.3 per cent per year in the next three decades. In part due to volatile oil prices, Alberta's GDP has been very volatile, with 10-, 20- and 30-year average real per capita growth rates of 3.8, 3.8 and 1.4 per cent per year, respectively. In part, these growth rates reflect the expansion of the resource-extraction sector. To calculate growth in non-resource GDP, we calculate resource rents as described above. Figure B.3 shows the share of resource revenues and resource rents of total Alberta GDP with averages of 24 per cent and 12 per cent for the range for which data is available. Subtracting resource rents gives per capita non-resource GDP growth rates of 3.2 per cent per year for the last 10 years and 3.0 per cent per year for the period for which we have calculated resource rents (17 years). Excluding not just resource rents, but total resource revenues we obtain 3.8 and 3.2 per cent per year, respectively.

We set trend population growth to $n = 1.3$ per cent per year with population size at end of 2012 equal to 3,873,745[25] and set trend growth of non-resource GDP to $n + g = 3.3$ per cent per year, which implies a trend productivity growth of $g = 2.0$ per cent per year.

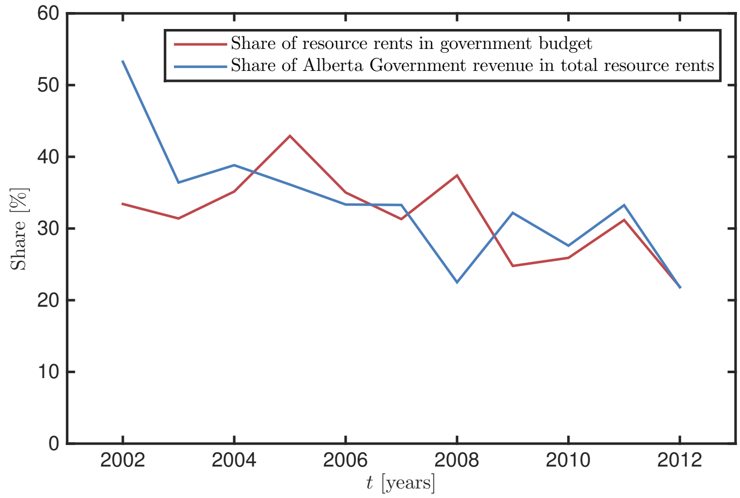
[23] cf. CERI, "Canadian oil sands."
[24] Statistics Canada, Cansim.
[25] ibid.

Figure B.4: Alberta government resource revenues.

(a) Absolute resource revenues and rents.



(b) Resource revenues and rents as a share.



B.7 Historical series of government resource rents

Alberta government income derived from the extraction of oil and gas and bitumen consists of a variety of fees and royalties of which the conventional oil royalty, the oil sands royalty and the natural gas and by-product royalty are the major components. In addition, the Alberta government receives a share of the corporate income tax paid by the resource-extracting sector. Figure B.4 shows the sum of these rents as received by the government in absolute values[26] and as a share of total resource rents.

On average, resource revenues constitute approximately one-third of total Alberta government revenues (part of of the income from corporate income taxation is received at a national level by the federal government). The government take (the share of total resource rents that is ultimately received by the Alberta government) is 34 per cent for the period 2002-2012. To estimate optimal savings for the Alberta government, we take optimal savings for the economy as a whole and multiply it by 0.34. The government can only save that part of resource rents that it receives as royalty or tax income in the first place. We also suppose in our calculations that the size of the Alberta government relative to the total Alberta economy stays constant at 14 per cent, based on an historical average.

B.8 Initial size of the fund

We include the Contingency Account ($2.7 billion) and the Heritage Savings Trust Fund ($14.9 billion) to give a total initial fund size of $17.6 billion (5.7 per cent of total GDP in 2012) at the end of March 2013.[27] We do not include the much smaller funds, such as the Medical Research Endowment Fund and the Scholarship Fund, since these have not been funded by oil and gas revenues and reflect savings as part of the non-resource part of the economy.

[26] Government of Alberta, Treasury Board and Finance, Private Communication (2013).

[27] Government of Alberta, Budget 2013, `http://budget2013.alberta.ca/`.

Appendix C

Appendix to Chapter 4

C.1 Valuing oil with exogenous oil extraction

Asset returns are assumed to be normally distributed and can be expressed as a linear combination of m independent shocks, $dZ = \Lambda^* du^*$ where du^* is an $m \times 1$ vector. If the oil price is completely spanned by the market, it can be expressed as a linear combination of these m shocks: $dZ_O = \Lambda_O^* du^*$. Now, in order to study incomplete markets, remove one asset from the investment set. The returns on the remaining $m-1$ assets can now be expressed as a linear combination of $m-1$ (different) independent shocks, $dZ = \Lambda du$ where du is an $(m-1) \times 1$ vector. If oil returns are expressed in terms of these shocks, there is a residual part that is uncorrelated with the market, $dZ_O = \lambda_{Oh} du_h + \Lambda_O du$, as in (4.2.5). Although the asset that is removed from the investment set may be correlated with other assets, the unhedgeable component of the oil price is not. Here, we are concerned with valuing subsoil oil and so ignore investment restrictions. Any asset that is outside the investment set can still be observed and can be used to value oil wealth. The value thus derived is a market value. Taking equation (4.2.5) with $\lambda_{Oh} = 0$, we can express the oil price as:

$$P_O(t) = P_O(0) \exp(-\phi t) \prod_{i=1}^{m} \left[\frac{P_i(t)}{P_i(0)} \right]^{\beta_i}, \tag{C.1.1}$$

with $\phi \equiv -\alpha_0 + \sum\limits_{i=1}^{m} \beta_i \left(\alpha_i - \frac{1}{2}\sigma_i^2 \right) + \frac{1}{2} \sum\limits_{i=1}^{m} \sum\limits_{j=1}^{m} \beta_i \beta_j \sigma_{ij}$ and $\beta_i \equiv \sigma_O M_i / \sigma_i$, $M_i \equiv \left[\Lambda_O \Lambda^{-1} \right]_i$, which can be readily verified using Itō's lemma and comparing coefficients with (4.2.4).

Result 1: With complete markets, the capitalized value of oil income is:

$$V(P_O, t) = P_O(t) O(t) / \psi, \quad \psi \equiv r + \kappa - \alpha_O + \sum\nolimits_{i=1}^{m} \beta_i (\alpha_i - r). \tag{C.1.2}$$

Derivation: First, we construct a portfolio with value V $(P_1, .., P_m, t)$ which consists of assets $1, ..., n$ that is identical to the capitalized value of oil and distributes an amount of cash equal to per unit time. This value evolves according to:

$$dV = (\mu_V V - P_O O)dt + \sigma_V V dZ_V. \tag{C.1.3}$$

With the aid of Itō's lemma, the dynamics of the portfolio can be written as:

$$\begin{gathered} dV = \sum_{i=1}^{m} V_i dP_i + V_t dt + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} V_{ij} dP_i dP_j = \\ \left[\sum_{i=1}^{m} \alpha_i V_i P_i + V_t + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij} V_{ij} P_i P_j\right] dt + \sum_{i=1}^{m} \sigma_i V_i P_i dZ_i, \end{gathered} \tag{C.1.4}$$

where $V_i = \partial V/\partial P_i$ and $\sigma_{ij} = \sigma_i \sigma_j \rho_{ij}$. Comparing coefficients with (C.1.3) gives:

$$\mu_V V - P_O O = \sum_{i=1}^{m} \alpha_i P_i V_i + V_t + \frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m} \sigma_{ij} V_{ij} P_i P_j, \quad \sigma_V V dZ_V = \sum_{i=1}^{m} \sigma_i P_i V_i dZ_i. \tag{C.1.5}$$

Finally, let $dZ_V = \Lambda_V du$. This implies:

$$\sigma_V dZ_V = \sigma_V \Lambda_V du = \Gamma' dZ = \Gamma' \Lambda du, \quad \Gamma \equiv [V_1 \sigma_1 P_1 / V, .., V_m \sigma_m P_m / V]. \tag{C.1.6}$$

Second, we create another portfolio with value $X(t)$ that consists of oil wealth $V(t)$, the risky assets and the safe asset. This portfolio is dynamically constructed, so short positions offset long positions, there is no net risk, and the net value of the portfolio is always zero. Hence, the weight of the safe asset in total wealth is $w_r = -w_V - \sum_{i=1}^{m} w_i$, where w_V is the weight of oil in total wealth. The return to this portfolio is:

$$\begin{aligned} dX &= w_V\left(\frac{dV + P_O O dt}{V}\right) + \sum_{i=1}^{m} w_i\left(\frac{P_i}{P}\right) + w_r r dt \\ &= \left[w_v(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r)\right] dt + w_V \sigma_V dZ_V + \sum_{i=1}^{m} w_i \sigma_i dZ_i \\ &= \left[w_v(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r)\right] dt + w_V \Gamma' \Lambda du + \Psi \Lambda du, \end{aligned} \tag{C.1.7}$$

where the second equality follows from (C.1.3), the third equality from (C.1.6) and $\Psi \equiv [w_1\sigma_1, .., w_m\sigma_m]'$. Suppose that the weights in this new portfolio are dynamically constructed so that there is no risk: $w_V \Gamma' \Lambda du + \Psi \Lambda du = 0$ and the last two terms in the last equality of (C.1.7) vanish. The weights that would achieve this are $w_i = -(V_i/V)P_i w_V, \ i = 1, .., m$. Arbitrage dictates that such a constructed portfolio must have a zero expected excess return over the risk-free rate:

$$0 = w_V(\mu_V - r) + \sum_{i=1}^{m} w_i(\alpha_i - r), V(\mu_v - r) = \sum_{i=1}^{m} V_i P_i(\alpha_i - r). \tag{C.1.8}$$

Combining (C.1.8) with (C.1.5) gives the following optimality condition:

$$\frac{1}{2}\sum_{i=1}^{m}\sum_{j=1}^{m}\sigma_{ij}P_iP_jV_{ij}+\sum_{i=1}^{m}rP_iV_i-rV+V_t+P_OO=0. \qquad \text{(C.1.9)}$$

Third, the proposed capitalized value of oil income and associated partials,

$$\begin{aligned}&V(P_O,t)=\tfrac{1}{\psi}P_O(0)\exp(-\phi t)\prod_{i=1}^{m}\left[\tfrac{P_i(t)}{P_i(0)}\right]^{\beta_i}O(0)\exp(-\kappa t),\quad V_i=\tfrac{\beta_iV}{P_i},\\ &V_t=-\phi V,\quad V_{ii}=\tfrac{\beta_i(\beta_i-1)V}{P_i^2},\quad V_{ij}=\tfrac{\beta_i\beta_jV}{P_iP_j},\ j=1,..,m,\quad i=1,..,m,\end{aligned} \qquad \text{(C.1.10)}$$

satisfy (C.1.9) by substitution. Result 1 thus gives capitalized oil income.

Result 1 establishes (4.3.1). The instantaneous rate of change in the value of oil income is found by applying Itō's lemma to this equation to give:

$$P_OOdt+dV=\left[r+\sum_{i=1}^{m}\beta_i(\alpha_i-r)\right]Vdt+\sigma_OVdZ_O. \qquad \text{(C.1.11)}$$

The result in (4.3.2) follows from substituting (C.1.11), (4.2.2), and (4.2.5) with $\lambda_{Oh}=0$ into the expression for total wealth, $dW=dF+dV$. With an investment restriction on asset m, the derivation for the value of the windfall is analogous and (C.1.1) still holds. Asset m is then replaced by the unspanned component of the oil price h and $\beta_h=(\sigma_O/\sigma_h)\,\lambda_{Oh}$.

C.2 Asset allocation with exogenous oil extraction

Here we derive the optimal portfolio weights in a sovereign wealth fund in the presence of oil, with and without investment restrictions based on Merton (1990). We begin by restricting investment in asset h, so $\lambda_{Oh}\neq 0$ and the fund holds $m-1$ securities. The unspanned component of the oil price is uncorrelated with other assets:

$$dP_h=\alpha_hP_hdt+\sigma_hP_hdu_h, \qquad \text{(C.2.1)}$$

Note that m was a traded asset that was correlated with all other assets. Above-ground wealth is accumulated according to (4.2.2). We obtain:

$$\begin{aligned}dF&=\sum_{i=1}^{m-1}w_iF(\alpha_i-r)dt+(rF+P_OO-C)\,dt+\sum_{i=1}^{m-1}w_iF\sigma_idZ_i\\ dV+P_OOdt&=\left(r+\sum_{i=1}^{m-1}\beta_i(\alpha_i-r)+\beta_h(\alpha_h-r)\right)Vdt\\ &\quad+\sigma_OV(MdZ+\lambda_{Oh}du_O)\,.\end{aligned} \qquad \text{(C.2.2)}$$

The Hamilton-Jacobi-Bellman (HJB) equation is:

$$\max_{w_i,C}\left[U(C)\,e^{-\rho t}+\tfrac{1}{dt}E_t\left[dJ(F,V,t)\right]\right]=0, \qquad \text{(C.2.3)}$$

$$
\begin{aligned}
\tfrac{1}{dt}E_t\left[dJ\right] = J_t + J_F\left(\sum_{i=1}^{m-1} w_i F\left(\alpha_i - r\right) + rF + P_O O - C\right) \\
+J_{VF} VF \sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i \beta_j \sigma_{ij} \\
+J_V\left(\left(r + \sum_{i=1}^{m-1}\beta_i\left(\alpha_i - r\right) + \beta_h\left(\alpha_h - r\right)\right)V - P_O O\right) \\
+\tfrac{1}{2}J_{FF}F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j \sigma_{ij} + \tfrac{1}{2}J_{VV}V^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{O,0}^2\right].
\end{aligned}
\tag{C.2.4}
$$

The first-order conditions with respect to C and w_i are:

$$
U'\left(C\right)e^{-\rho t} - J_F = 0 \Rightarrow J_F = U'\left(C\right)e^{-\rho t} \tag{C.2.5}
$$

$$
J_F F\left(\alpha_i - r\right) + J_{FF}F^2\sum_{j=1}^{m-1} w_j\sigma_{ij} + J_{FV}FV\sum_{j=1}^{m-1}\beta_j\sigma_{ij} = 0. \tag{C.2.6}
$$

Equation (C.2.6) gives the optimal weights in the fund:

$$
\begin{aligned}
w_i &= -\frac{J_F}{FJ_{FF}}\sum_{j=1}^{m-1}\nu_{ij}\left(\alpha_j - r\right) - \frac{J_{FV}}{J_{FF}}\frac{V}{F}\beta_i \\
&= \frac{C/F}{\partial C/\partial F}\theta\sum_{j=1}^{m-1}\nu_{ij}\left(\alpha_j - r\right) - \frac{\partial C/\partial V}{\partial C/\partial F}\frac{V}{F}\beta_i.
\end{aligned}
\tag{C.2.7}
$$

If markets are complete, $\partial C/\partial F = \partial C/\partial V = \partial C/\partial W = MPC$ from (4.3.7). If markets are incomplete, instead of solving the arising partial differential equations numerically, we approximate these partials from the complete markets case or, alternatively, assume that consumption is a linear function of total wealth. With and without investment restrictions we then obtain:

$$
w_i = \frac{W}{F}\theta\sum_{j=1}^{m-1}\nu_{ij}\left(\alpha_j - r\right) - \frac{V}{F}\beta_i. \tag{C.2.8}
$$

Defining $\bar{w}_i W \equiv w_i F + \beta_i V$, rearranging (C.2.8) gives (4.3.3) and (4.3.5).

C.3 Optimal consumption with exogenous oil extraction

If markets are complete we can find a closed-form solution for the value function $J(F, V, t) = J(W \equiv F + V, t)$ from Merton (1990). Substituting the first-order conditions (C.2.5) and (C.2.6) into the HJB equation (C.2.3) gives:

$$
0 = \tfrac{1}{\theta-1}\exp(-\theta\rho t)J_W^{1-\theta} + J_t + rWJ_W - \frac{J_W^2}{J_{WW}}\frac{(\alpha_W - r)^2}{2\sigma_W^2}. \tag{C.3.1}
$$

The closed-form solution to this stochastic partial differential equation is:

$$
J(W, t) = \tfrac{\theta}{\theta-1}\exp(-\rho t)[\theta\rho - (\theta - 1)\eta]^{-1/\theta}W^{(\theta-1)/\theta}, \quad \eta = r + \theta(\alpha_W - r)^2/2\sigma_W^2. \tag{C.3.2}
$$

Equation (4.3.7) follows from substituting (C.3.2) into (C.2.5). Applying Itō's lemma to (C.2.5):

$$\frac{\frac{1}{dt}E_t[dJ_F]}{J_F} = \frac{C''(C)}{U'(C)}\frac{\frac{1}{dt}E_t[dC]}{C} - \rho + \frac{1}{2}\frac{CU'''(C)}{U'(C)}\frac{\frac{1}{dt}E_t[dC^2]}{C}. \tag{C.3.3}$$

Using Itō's lemma we obtain:

$$\begin{aligned} dJ_F(F,V,t) = J_{FF}dF + J_{FV}dV + J_{Ft}dt + \tfrac{1}{2}J_{FFF}dF^2 \\ +\tfrac{1}{2}J_{FVV}dV^2 + J_{FFV}dFdV. \end{aligned} \tag{C.3.4}$$

In addition the derivative of (C.2.3) with respect to F is:

$$\begin{aligned} 0 = J_{tF} + J_{FF}\left(\sum_{i=1}^{m-1} w_i F(\alpha_i - r) + rF + P_O O - C\right) \\ +J_F\left(\sum_{i=1}^{m-1} w_i(\alpha_i - r) + r\right) + J_{VF}V\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij} \\ +J_{VF}\left(\left(r + \sum_{i=1}^{m-1}\beta_i(\alpha_i - r) + \beta_h(\alpha_h - r)\right)V - P_O O\right) \\ +\tfrac{1}{2}J_{FFF}F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j\sigma_{ij} + J_{FF}F\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j\sigma_{ij} \\ +\tfrac{1}{2}J_{VVF}V^2\left[\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{Oh}^2\right] + J_{VFF}VF\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij}. \end{aligned} \tag{C.3.5}$$

Substituting (C.2.6) and (C.3.4) into (C.3.5) gives:

$$0 = \tfrac{1}{dt}E_t[dJ_F] + J_F r. \tag{C.3.6}$$

We also have:

$$\tfrac{1}{dt}E_t\left[(dC)^2\right] = C_F^2\tfrac{1}{dt}E_t\left[(dF)^2\right] + C_V^2\tfrac{1}{dt}E_t\left[(dV)^2\right] + 2C_V C_F\tfrac{1}{dt}E_t[dVdF], \tag{C.3.7}$$

$$\begin{aligned} \tfrac{1}{dt}E_t\left[(dF)^2\right] &= F^2\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i w_j\sigma_{ij}, \\ \tfrac{1}{dt}E_t[dVdF] &= VF\sum_{i=1}^{m-1}\sum_{j=1}^{m-1} w_i\beta_j\sigma_{ij}, \\ \tfrac{1}{dt}E_t\left[(dV)^2\right] &= V^2\left(\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\beta_i\beta_j\sigma_{ij} + \sigma_O^2\lambda_{Oh}^2\right). \end{aligned} \tag{C.3.8}$$

Combining (C.3.7) and (C.3.8), we obtain:

$$\begin{aligned} \tfrac{1}{dt}E_t[dC^2] = C_W^2\left[\textstyle\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}(w_i F + \beta_i V)(w_j F + \beta_j V)\sigma_{ij} + \lambda_{Oh}^2\sigma_O^2 V^2\right] \\ = C_W^2\left(\textstyle\sum_{i=1}^{m-1}\sum_{j=1}^{m-1}\bar{w}_i\bar{w}_j\sigma_{ij}W^2 + \lambda_{Oh}^2\sigma_O^2 V^2\right) = C_W^2\left(\bar{w}^2\sigma_W^2 W^2 + \lambda_{Oh}^2\sigma_O^2 V^2\right), \end{aligned} \tag{C.3.9}$$

where we use $C_W \approx C/W$, $C_W \approx C_F \approx C_V$. The stochastic Euler equations (4.3.6) and (4.4.1) follow from substituting (C.3.6) and (C.3.9) into (C.3.3). Equation (4.3.6) assumes complete markets, so $\lambda_{Oh} = 0$ and $C_W = C/W$ from (4.3.7). The result is exact. Equation (4.4.1) assumes incomplete markets, so $\lambda_{Oh} \neq 0$ and we use $C_W \approx C/W$ in order to obtain an approximate solution in the absence of a closed-form one.

C.4 Complete markets and exogenous oil paths: EZ preferences

The results in §4.4 follow from solving the HJB equation in the undiscounted value function $J(F)$ modified for Epstein-Zin preferences (Duffie and Epstein 1992):

$$\begin{aligned} 0 \quad = \quad \max_{w,C} \Bigg\{ & \frac{\rho}{1-1/EIS} \frac{C^{1-1/EIS} - ((1-CRRA)J(F))^{\frac{1-1/EIS}{1-CRRA}}}{[(1-CRRA)J(F)]^{\frac{CRRA-1/EIS}{1-CRRA}}} + ... \\ & + J'(F)\,[rF - C + w(\alpha - r)F] + \tfrac{1}{2}J''(F)w^2\sigma^2F^2 \Big\}. \end{aligned} \tag{C.4.1}$$

It can be verified that (C.4.1) has the solution:

$$J(F) = \frac{(\Theta F)^{1-CRRA}}{1-CRRA}, \Theta = \left(\rho \times EIS + \left[r + (\alpha - r)^2 \Big/ \left(2\sigma^2 \times CRRA\right)\right](1 - EIS)\right)^{\frac{1}{1-EIS}} \rho^{\frac{EIS}{EIS-1}}. \tag{C.4.2}$$

C.5 Endogenous oil extraction

The HJB equation for the problem in (4.5.1), (4.5.2), (4.2.3), (4.2.4) and (4.5.3) is:

$$\begin{aligned} 0 = \max_{C,w_i,O} & \left\{ U(C)e^{-\rho t} + \tfrac{1}{dt}E_t\,[dJ(F, P_O, S, t)] \right\}, \\ \tfrac{1}{dt}E_t\,[dJ(F,P_O,S,t)] = J_F & \left[\textstyle\sum_{i=1}^m w_i(\alpha_i - r)F + rF - C + P_O O - G(O)\right] + J_P\alpha_O P_O \\ -J_S O + J_t + \tfrac{1}{2}J_{FF}F^2 & \textstyle\sum_{i=1}^m \sum_{j=1}^m w_i w_j \sigma_{ij} + \tfrac{1}{2}J_{PP}\sigma_O^2 P_O^2 + J_{FP}\sigma_O P_O F \sum_{i=1}^m w_i\sigma_i\rho_{iO}. \end{aligned} \tag{C.5.1}$$

The first-order conditions are:

$$U'(C)e^{-\rho t} = J_F, \tag{C.5.2}$$

$$J_F F(\alpha_i - r) + J_{FF}F^2 \sum_{j=1}^m w_j\sigma_{ij} + J_{FP}F\sigma_i\sigma_O\rho_{iO}P_O = 0 \quad \forall i, \tag{C.5.3}$$

$$J_F\,[P_O - G'(O)] - J_S = 0. \tag{C.5.4}$$

Differentiating (C.5.1) with respect to the states invoking the envelope theorem

$$\tfrac{1}{dt}E\,[dJ_F] + J_F\left[r + \sum_{i=1}^m w_i(\alpha_i - r)\right] + J_{FF}F\sum_{i=1}^m\sum_{j=1}^m w_i w_j\sigma_{ij} + J_{FP}\sigma_O P_O\sum_{i=1}^m w_i\sigma_i\rho_{iO} = 0, \tag{C.5.5}$$

$$\tfrac{1}{dt}E\left[dJ_S\right] = 0, \tag{C.5.6}$$

$$\tfrac{1}{dt}E\left[dJ_P\right] + J_F O + J_P \alpha_O + J_{PP}\sigma_O^2 P_O + J_{FP} F \sum_{i=1}^{m} w_i \sigma_i \sigma_O \rho_{iO} = 0. \tag{C.5.7}$$

Upon substitution of (C.5.3) into (C.5.5), we obtain:

$$\tfrac{1}{dt}E\left[dJ_F\right] = -rJ_F. \tag{C.5.8}$$

Applying Itō's lemma to (C.5.2) and combining the result with (C.5.8) gives:

$$\frac{\tfrac{1}{dt}E_t\left[dC\right]}{C} = \left[\frac{-U'(C)}{CU''(C)}\right](r-\rho) - \left[\frac{CU'''(C)}{U''(C)}\right]\frac{\tfrac{1}{dt}E_t\left[dC^2\right]}{C^2}. \tag{C.5.9}$$

Applying Itō's lemma to (C.5.4) gives:

$$\frac{dJ_S}{J_S} = \frac{dJ_F}{J_F} + \frac{d\Omega_O}{\Omega_O} + \frac{dJ_F d\Omega_O}{J_F \Omega_O}, \quad \Omega_O = P_O - G'(O). \tag{C.5.10}$$

Combining (C.5.6), (C.5.8) and (C.5.10) yields (4.5.4). Itō's lemma yields:

$$dJ_F = -rJ_F dt + J_{FF} F \sum_{i=1}^{m} w_i \sigma_i dZ_i + J_{FP} P_O \sigma_O dZ_O, \tag{C.5.11}$$

$$dO = \mu_O(F, P_O, S, t)dt + O_F F \sum_{i=1}^{m} w_i \sigma_i dZ_i + O_P P_O \sigma_O dZ_O, \tag{C.5.12}$$

where use (C.5.8) and $\mu_O(F, P_O, S, t) = (1/dt)\, E_t\left[dO\right]$ is the to be determined expected rate of growth of the rate of oil extraction. Applying Itō's lemma to $\Omega_O = P_O - G'(O) = P_O - \gamma O$ gives:

$$d\Omega_O \;=\; dP_O - G''(O)dO - \tfrac{1}{2}G'''(O)dO^2 \tag{C.5.13}$$

with $G'''(O) = 0$ for quadratic costs. Multiplying (C.5.11) and (C.5.13) gives:

$$\begin{aligned}\tfrac{dJ_F d\Omega_O}{J_F \Omega_O} = &-\tfrac{\gamma O_F F}{J_F \Omega_O}\left\{\textstyle\sum_{i=1}^{m} w_i \sigma_i \left[J_{FP}\sigma_O P_O \rho_{iO} + J_{FF} F \sum_{j=1}^{m} w_j \sigma_j \rho_{ij}\right]\right\} dt \\ &+\tfrac{(1-\gamma O_P)P_O \sigma_O}{J_F \Omega_O}\left(J_{FP} P_O \sigma_O + J_{FF} F \textstyle\sum_{i=1}^{m} w_i \sigma_i \rho_{iO}\right) dt.\end{aligned} \tag{C.5.14}$$

Substituting (C.5.3) for all assets (the first term on the right-hand side) and for the perfectly correlated asset k ($\rho_{kO} = 1$, the second term) gives:

$$\frac{dJ_F d\Omega_O}{J_F \Omega_O} = \frac{1}{\Omega_O}\left[\gamma O_F F \sum_{i=1}^{m} w_i(\alpha_i - r) - (1-\gamma O_P)\left(\frac{\alpha_k - r}{\sigma_k}\right) P_O \sigma_O\right] dt. \tag{C.5.15}$$

Substituting (C.5.13) and (C.5.15) into (4.5.4) gives:

$$d\Omega_O = r\Omega_O dt - \gamma O_F \sum_{i=1}^{m}\left[(\alpha_i - r)dt + \sigma_i dZ_i\right] w_i F + (1-\gamma O_P)\left[(\alpha_k - r)dt + \sigma_k dZ_O\right] P_O \frac{\sigma_O}{\sigma_k}. \tag{C.5.16}$$

Result 2: If all prices are deterministic, $\dot{\Omega}_O = r\Omega_O$. If the oil price is also without drift, $\alpha_O = 0$, the date of exhaustion is $T = -\frac{1}{r}\ln(\gamma O(0)/P_O(0))$ and the optimal rate of oil extraction $O(t)$ is to leading-order approximation:

$$O(t) = \sqrt{2\tfrac{r}{\gamma}S(t)P_O(t)}. \tag{C.5.17}$$

Derivation: Using $\Omega_O = P_O - \gamma O$ in the deterministic Hotelling rule, we get $\dot{O} = rO + \frac{1}{\gamma}(\alpha_O - r)P_O(0)e^{\alpha_O t}$ which can be solved to give:

$$O(t) = O(0)e^{rt} + \tfrac{1}{\gamma}P_O(0)\left(e^{\alpha_O t} - e^{rt}\right). \tag{C.5.18}$$

We exclude $\alpha_O \geq r$ as price growth would delay extraction indefinitely. Provided $\Omega_O(0) > 0$ and, $\alpha_O < r$, the extraction rate remains finite. The optimal initial extraction rate satisfies, $S(t) = \int_t^T O(\tau)d\tau$, and the date of exhaustion T must satisfy $O(T) = 0$. The date of exhaustion only has an explicit solution if $\alpha_O = 0$:

$$T = -\tfrac{1}{r}\ln(1 - R), \tag{C.5.19}$$

$$S(0) = -\tfrac{1}{r\gamma}P_O(0)\left(\ln(1 - R) + R\right), \tag{C.5.20}$$

where $0 < R = \gamma O(0)/P_O(0) < 1$ is small. As (C.5.20) only defines the initial rate of extraction, $O(0) = f(S(0), P_O(0))$, we use asymptotic methods to find a series-solution and get the leading-order effect. Since $-\ln(1 - R) = \sum_{n=1}^{\infty}\frac{1}{n}R^n$ we obtain:

$$\frac{r\gamma S(0)}{P_O(0)} = \sum_{n=2}^{\infty}\frac{R^n}{n}. \tag{C.5.21}$$

This can be inverted to give:

$$O(t) = S(t)\left[\sqrt{2}\xi(t)^{-1} - \tfrac{2}{3} + \tfrac{1}{9\sqrt{2}}\xi(t) + \tfrac{2}{135}\xi(t)^2 + \tfrac{1}{540\sqrt{2}}\xi(t)^3 + o(\xi(t)^4)\right], \tag{C.5.22}$$

where $\xi(t) = \sqrt{r\gamma S(t)/P_O(t)}$ and the coefficients stem from the series inversion and are independent of parameters. The leading order yields (C.5.17). Result 2 gives:

$$\partial O/\partial F = 0, \qquad \partial O/\partial P_O = O/(2P_O) = \sqrt{rS/(2\gamma P_O)}. \tag{C.5.23}$$

Assuming the effect of uncertainty is modest, we use these partial derivatives in the analogous problem (analogous to taking the leading-order terms in a perturbation expansion in the volatility of the oil price). Substituting these partials into (C.5.16) gives:

$$d\Omega_O = r\Omega_O dt + (P_O - \gamma\tfrac{1}{2}O)\frac{\sigma_O}{\sigma_k}\left[(\alpha_k - r)dt + \sigma_k dZ_O\right]. \tag{C.5.24}$$

Combining (C.5.13) and (C.5.24) and setting $\alpha_O = 0$ as in (A46) gives:

$$\tfrac{1}{dt}E[dO] = \mu_O(F, P_O, S, t) = -\tfrac{1}{\gamma}\left(r + \tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)P_O + \left(r + \tfrac{1}{2}\tfrac{\sigma_O}{\sigma_k}(\alpha_k - r)\right)O. \tag{C.5.25}$$

Equation (4.5.5) is found by substituting (C.5.25) into (C.5.12) and solving the initial value problem subject to the exhaustion condition $O(t = T) = S(t = T) = 0$. To obtain the results in Figure 4.5, we use the full series solution in (C.5.22) , which becomes exact in the limit of an infinite number of terms.

C.6 Asset allocation with endogenous oil extraction

Endogenous oil rents can be replicated with a bundle of N_k shares of asset k and N_r shares of the safe asset, $X \equiv N_k \ P_k + N_r \ P_r$. This replicating bundle can be constructed as follows. To finance the dividend, the price must increase or shares must be sold:

$$- \Omega dt = \sum_{i=k,r} dN_i dP_i + dN_i P_i \tag{C.6.1}$$

Equation (4.5.6) combines this expression for the dividends with the path for the replicating bundle. By Itō's lemma the replicating bundle must satisfy:

$$\begin{aligned} dX + \Omega dt &= \textstyle\sum_{i=k,r} (N_i dP_i + dN_i dP_i + dN_i P_i) + \Omega dt \\ &= \textstyle\sum_{i=k,r} (N_i dP_i) = \omega_k X(\alpha_k - r)dt + rXdt + \omega_k X \sigma_k dZ_k. \end{aligned} \tag{C.6.2}$$

where $\omega_k(t) \equiv N_k(t)P_k(t)/X(t)$. The weights $\omega_k(t)$ are updated continuously to match the stochastic path of oil rents (C.5.16). As oil wealth and the replicating bundle have the same properties they must also have the same value, $X = V$, giving (4.5.6). We have focused on $dV(t) + \Omega(t)dt$. $V(t)$ is found using contingent claims (Merton 1990) if oil rents follow the Itō process $d\Omega(t) = a(.)\Omega dt + s(.)\Omega dZ_O$ and $a(.)$ and $s(.)$ are not constants. The value of oil rents must be that of the replicating bundle, $V(t) = X(t)$. Equation (4.5.7) states that the problem can be summarized in terms of $W(t) = F(t) + V(t)$. Combining (4.5.2) and (C.6.2) gives (4.5.7). The weight of asset k in the fund adjusts continuously so that the net weight of oil in total wealth is constant. The weight of all other assets in the fund remains constant, as in (4.5.8).

Appendix D

Appendix to Chapter 5

D.1 Risk-adjusted carbon price with convex reduced-form damages

We can generalize Result 2 to convex reduced-form damages, skewed damage uncertainty and carbon stock uncertainty (see Appendix D.5 for the derivation). The resulting rule includes additional correction factors, which can be evaluated as simple, one-dimensional integrals along a business-as-usual path. The only additional assumption is that the future atmospheric carbon stock does not inherit any of the uncertainty from new emissions through their dependence on the stochastic capital stock (cf. (D.5.4)), which is only associated with a very small error, as evident from Section 5.6.4 and Appendix D.7.

Result 2′: In general ($\theta_{ET} \neq 0$, $\theta_\lambda \neq 0$ and $\sigma_E > 0$), the leading-order optimal SCC is:

$$P = \frac{\mu\ \Theta(E)\ Y|_{P=0}}{r^\star} \left(1 + \theta_{ET} \frac{\mu F^{(0)}}{E} \frac{\Upsilon}{r^{\star\star}} + \Delta_{EE} + \Delta_{\chi\chi} + \Delta_{\lambda\lambda} + \Delta_{CK} + \Delta_{CC}\right), \tag{D.1.1}$$

where we redefine

$$r^\star \equiv \rho + (\gamma - 1)\left(g^{(0)} - \frac{1}{2}\eta\sigma_K^2\right) + (1 + \theta_{ET})\varphi \tag{D.1.2}$$

to include a more general dependence on φ, $r^{\star\star} \equiv r^\star + (\eta - 1)\sigma_K^2 - \varphi$, and $F^{(0)}$ is shorthand for optimal fossil fuel use without climate policy,

$$F^{(0)} = ((1 - \alpha)/b)^{\frac{1}{\alpha}} A^{\frac{1}{\alpha}} K, \tag{D.1.3}$$

to the zeroth order of approximation. The deterministic correction factor for future emissions and $\theta_{ET} \neq 0$ is

$$\Upsilon = \frac{r^{\star\star}}{1-(1+\theta_{ET})\varphi/r^{\star}} \times \int_0^{\infty} \left(\exp(-r^{\star\star}s) - \frac{(1+\theta_{ET})\varphi}{r^{\star}} \exp(-(r^{\star}-\varphi)s) \right) e(s)^{\theta_{ET}-1} ds, \quad \text{(D.1.4)}$$

where $e(s) \equiv 1 + (\mu F^{(0)}/E)(\exp(\varphi s) - 1)/\varphi$ captures new emissions and s is the dummy variable of integration. We also have the corrections for uncertainty in the carbon stock, climate sensitivity and damages, which are now multiplied by new correction factors

$$\Delta_{EE} = \frac{1}{2}\theta_{ET}(1-\theta_{ET})\left(\frac{\sigma_E}{E}\right)^2 \frac{1}{r^{\star}-2\varphi}\Upsilon_{EE}, \quad \text{(D.1.5)}$$

$$\Delta_{\chi\chi} = \frac{1}{2}\theta_{\chi T}(1+\theta_{\chi T})\frac{\left(\sigma_{\chi}/\bar{\chi}\right)^2}{r^{\star}+2\nu_{\chi}}\Upsilon_{\chi\chi} \text{ and } \Delta_{\lambda\lambda} = \frac{1}{2}\theta_{\lambda}(1+\theta_{\lambda})\frac{\left(\sigma_{\lambda}/\bar{\lambda}\right)^2}{r^{\star}+2\nu_{\lambda}}\Upsilon_{\lambda\lambda} \quad \text{(D.1.6)}$$

with

$$\begin{aligned}\Upsilon_{ij} = 1 + \theta_{ET}\frac{\mu F^{(0)}}{E}\frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} & \\ \times\Bigg[\left(1+\frac{(1+\theta_{ET})\varphi}{\nu_i+\nu_j}\right)\int_0^{\infty}\exp\left(-(r^{\star}+\nu_i+\nu_j-\varphi)s\right)e(s)^{\theta_{ET}-1}ds & \\ -\frac{(1+\theta_{ET})\varphi}{\nu_i+\nu_j}\frac{r^{\star}+\nu_i+\nu_j}{r^{\star}}\int_0^{\infty}\exp(-(r^{\star}-\varphi)s)e(s)^{\theta_{ET}-1}ds + \frac{r^{\star}+\nu_i+\nu_j}{\nu_i+\nu_j} & \\ \times\int_0^{\infty}\left(\exp(-r^{\star\star}s)-\exp(-(r^{\star\star}+\nu_i+\nu_j)s)\right)e(s)^{\theta_{ET}-1}ds\Bigg] & \end{aligned} \quad \text{(D.1.7)}$$

for $i, j = \chi, \lambda$. The correction for correlated climate and economic risks is

$$\Delta_{\mathrm{CK}} = -(\eta-1)\sigma_K \times \left(\theta_{ET}\frac{\rho_{KE}\sigma_E}{(r^{\star}-\varphi)E}\Upsilon_{KE} + (1+\theta_{\chi T})\frac{\rho_{K\chi}\frac{\sigma_{\chi}}{\bar{\chi}}}{r^{\star}+\nu_{\chi}}\Upsilon_{K\chi} + (1+\theta_{\lambda})\frac{\rho_{K\lambda}\frac{\sigma_{\lambda}}{\bar{\lambda}}}{r^{\star}+\nu_{\lambda}}\Upsilon_{K\lambda}\right), \quad \text{(D.1.8)}$$

with

$$\begin{aligned}\Upsilon_{Ki} = 1 + \theta_{ET}\frac{\mu F^{(0)}}{E}\frac{1}{1-(1+\theta_{ET})\varphi/r^{\star}} & \\ \times\Bigg[\left(1+\frac{(1+\theta_{ET})\varphi}{\nu_i}\right)\int_0^{\infty}\exp\left(-(\nu_i+r^{\star}-\varphi)s\right)e(s)^{\theta_{ET}-1}ds & \\ -(1+\theta_{ET})\frac{\varphi}{r^{\star}}\frac{r^{\star}+\nu_i}{\nu_i}\int_0^{\infty}\exp\left(-(r^{\star}-\varphi)s\right)e(s)^{\theta_{ET}-1}ds & \\ +\frac{2-\eta}{1-\eta}\frac{\nu_i+r^{\star}}{\nu_i}\int_0^{\infty}\left(\exp(-r^{\star\star}s)-\exp(-(r^{\star\star}+\nu_i)s)\right)e(s)^{\theta_{ET}-1}ds\Bigg] & \end{aligned} \quad \text{(D.1.9)}$$

for $i = \chi, \lambda$. The correction for correlated climate sensitivity and damage risks is

$$\Delta_{\mathrm{CC}} = \theta_{ET}(1+\theta_{\chi T})\frac{\rho_{E\chi}\sigma_E\sigma_\chi/\bar{\chi}}{(r^\star+\nu_\chi)E}\frac{r^\star}{r^\star-\varphi}\Upsilon_{E\chi}$$

$$+ (1+\theta_\lambda)\left(\theta_{ET}\frac{\rho_{E\lambda}\sigma_E\sigma_\lambda/\bar{\lambda}}{(r^\star+\nu_\lambda)E}\frac{r^\star}{r^\star-\varphi}\Upsilon_{E\lambda} + (1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\bar{\chi}\bar{\lambda}}{r^\star+\nu_\chi+\nu_\lambda}\Upsilon_{\chi\lambda}\right), \quad \text{(D.1.10)}$$

where $\Upsilon_{\chi\lambda}$ is already defined in (D.1.7) and we do not show $\Upsilon_{E\chi}$ and $\Upsilon_{E\lambda}$. □

Due to the very small magnitude of $\sigma_E^2/g_0^{(0)}$ (see Section 5.4), we can ignore all terms involving atmospheric carbon stock volatility, as their contributions to the risk-adjusted SCC are negligible in which case $\Delta_{EE} = 0$ and (D.1.8) and (D.1.10) simplify to:

$$\Delta_{\mathrm{CK}} = -(\eta-1)\sigma_K\left((1+\theta_{\chi T})\frac{\rho_{K\chi}\sigma_\chi/\bar{\chi}}{r^\star+\nu_\chi}\Upsilon_{K\chi} + (1+\theta_\lambda)\frac{\rho_{K\lambda}\sigma_\lambda/\bar{\lambda}}{r^\star+\nu_\lambda}\Upsilon_{K\lambda}\right) \quad \text{(D.1.11)}$$

and

$$\Delta_{\mathrm{CC}} = (1+\theta_\lambda)(1+\theta_{\chi T})\frac{\rho_{\chi\lambda}\sigma_\chi\sigma_\lambda/\bar{\chi}\bar{\lambda}}{r^\star+\nu_\chi+\nu_\lambda}\Upsilon_{\chi\lambda}. \quad \text{(D.1.12)}$$

There are three differences compared to the simpler Result 2. First, convexity of reduced-form damages ($\theta_{ET} > 0$) pushes up the deterministic SCC though the deterministic correction factor $\Upsilon > 0$ in (D.1.4), but also boosts the discount rate $r^\star = \rho + (\gamma-1)g^{(0)} + (1+\theta_{ET})\varphi$. The net effect is ambiguous, but positive for small decay rates of atmospheric carbon. Second, our power-function specification for damages (5.2.6) gives

$$\Theta_E(E) = (1/S_{PI})^2\,(1+\theta_{ET})\theta_{ET}(E/S_{PI})^{\theta_{ET}-1} \quad \text{(D.1.13)}$$

to leading-order in our small parameter. With convex damages ($\theta_{ET} > 0$), the flow damage coefficient thus rises with the stock of atmospheric carbon. The time path for the carbon price is then steeper than that of world GDP. Third, there is a *climate damage uncertainty correction*

$$\Delta_{\lambda\lambda} = (1/2)\theta_\lambda(1+\theta_\lambda)(\sigma_\lambda/\bar{\lambda})^2\Upsilon_{\lambda\lambda}/(r^\star+2\nu_\lambda) \quad \text{(D.1.14)}$$

in (D.1.6), which adjusts the SCC upwards if the probability density function of damage shocks is right-skewed ($\theta_\lambda > 0$). The upward adjustment is larger if damages are more uncertain and right-skewed, display less mean reversion, and if the growth-corrected discount rate is smaller (higher σ_λ, θ_λ, lower ν_λ, and lower $r^\star$). This correction is separate from the negative effect on the risk-adjusted carbon price of the risk insurance term $\eta\sigma_K^2$, resulting from damages being proportional to GDP (as discussed in Section 5.3.2). Finally, the correction factors Υ_{ij} for $i, j = \chi, \lambda$ which appear in Result 2′ are unity for proportional damages ($\theta_{ET} = 0$), but are greater than unity for convex damages ($\theta_{ET} > 0$) and capture the contribution by new emissions.

D.2 Transformation to non-dimensional form

We define the non-dimensional variables

$$\begin{aligned}
&\hat{K} \equiv \frac{K}{K_0}, \quad \hat{E} \equiv \frac{E}{E_0}, \quad \hat{\chi} \equiv \frac{\chi}{\overline{\chi}}, \quad \hat{\lambda} \equiv \frac{\lambda}{\overline{\lambda}}, \quad \hat{F} \equiv \frac{F}{F_0}, \\
&\hat{C} \equiv \frac{C}{C_0}, \quad \hat{I} \equiv \frac{I}{C_0}, \quad \hat{\Phi} \equiv \frac{\phi}{C_0}, \quad \hat{t} \equiv g_0 t, \quad \hat{J} \equiv g_0 J/(C_0)^{1-\eta},
\end{aligned} \tag{D.2.1}$$

where zero subscripts refer to initial values ($t = 0$), except for

$$F_0 \equiv A\,(E_0)^{\frac{1}{\alpha}}\,((1-\alpha)\,/b)^{\frac{1}{\alpha}}\,K_0 \tag{D.2.2}$$

and $C_0 \equiv g_0 K_0$, so that all hatted quantities are $O(1)$ initially, assuming $\chi_0/\overline{\chi} = O(1)$ and $\lambda_0/\overline{\lambda} = O(1)$. We define $g_0 \equiv g\,(E = E_0)$ to be the growth rate of the economy without additional climate change, $\hat{\phi} \equiv \phi/g_0$ and $\hat{i} \equiv i/g_0$, where $i \equiv I/K$. The Hamilton-Jacobi-Bellman equation (5.3.2) becomes in non-dimensional terms

$$\begin{aligned}
0 = \max_{\hat{C},\hat{F}} \Bigg[&\frac{1}{1-\gamma} \frac{\hat{C}^{1-\gamma} - \hat{\rho}\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} + \hat{J}_{\hat{t}} + \hat{J}_{\hat{K}}\hat{\phi}(\hat{i})\hat{K} + \hat{J}_{\hat{E}}(\hat{\mu}\hat{F}e^{-\hat{g}\hat{t}} - \hat{\varphi}\hat{E}) \\
&+ \hat{J}_{\hat{\chi}}\hat{\nu}_{\chi}(1-\hat{\chi}) + \hat{J}_{\hat{\lambda}}\hat{\nu}_{\lambda}(1-\hat{\lambda}) + \frac{1}{2}\hat{J}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2 + \frac{1}{2}\hat{J}_{\hat{E}\hat{E}}\hat{\sigma}_E^2 + \frac{1}{2}\hat{J}_{\hat{\chi}\hat{\chi}}\hat{\sigma}_{\chi}^2 + \frac{1}{2}\hat{J}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^2 \Bigg] \\
&+ \hat{J}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{J}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + \hat{J}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda} + \hat{J}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} \\
&+ \hat{J}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \hat{J}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda},
\end{aligned} \tag{D.2.3}$$

where $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C} = \hat{A}(\hat{E},\hat{\chi})\hat{K}^{\alpha}\hat{F}^{1-\alpha} - \hat{b}\hat{F} - \hat{C}$, $\hat{Y} \equiv Y/C_0$ and $\hat{\phi} = \hat{i} - (1/2)\hat{\Omega}\hat{i}^2 - \hat{\delta}$. The resulting non-dimensional expressions are

$$\begin{aligned}
&\hat{\rho} \equiv \frac{\rho}{g_0}, \quad \hat{b} \equiv \frac{bF_0}{g_0 K_0}, \quad \hat{\omega} \equiv g_0\omega, \quad \hat{\delta} \equiv \frac{\delta}{g_0}, \quad \hat{g} \equiv \frac{g}{g_0}, \\
&\hat{\mu} \equiv \frac{\mu F_0}{g_0 E_0}, \quad \hat{\varphi} \equiv \frac{\varphi}{g_0}, \quad \hat{\nu}_{\chi} \equiv \frac{\nu_{\chi}}{g_0}, \quad \hat{\nu}_{\lambda} \equiv \frac{\nu_{\lambda}}{g_0}, \quad \hat{\sigma}_K \equiv \frac{\sigma_K}{\sqrt{g_0}}, \\
&\hat{\sigma}_E \equiv \frac{\sigma_E}{\sqrt{g_0}E_0}, \quad \hat{\sigma}_{\chi} \equiv \frac{\sigma_{\chi}}{\sqrt{g_0}\overline{\chi}} \quad \hat{\sigma}_{\lambda} \equiv \frac{\sigma_{\lambda}}{\sqrt{g_0}\overline{\lambda}},
\end{aligned} \tag{D.2.4}$$

with $\hat{A}\left(\hat{E},\hat{\chi}\right) \equiv A(E,\chi)F_0^{1-\alpha}/g_0 K_0^{1-\alpha}$. Damages and total factor productivity become

$$\begin{aligned}
\hat{D}(\hat{E},\hat{\chi},\hat{\lambda}) &\equiv \epsilon\hat{\lambda}^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}} \quad \text{and} \\
\hat{A} &\equiv \hat{A}^*(1-\hat{D}) = \hat{A}^*(1-\epsilon\hat{\lambda}^{1+\theta_\lambda}\hat{\chi}^{1+\theta_{\chi T}}\hat{E}^{1+\theta_{ET}}),
\end{aligned} \tag{D.2.5}$$

where the damage fraction $\hat{D} \equiv D$ is already non-dimensional,

$$\hat{A}^* \equiv A^* F_0^{1-\alpha}/g_0 K_0^{1-\alpha} \tag{D.2.6}$$

and the final non-dimensional parameter is

$$\epsilon \equiv \overline{\lambda}^{1+\theta_\lambda}\overline{\chi}^{1+\theta_{\chi T}}\left(\frac{E_0}{S_{PI}}\right)^{1+\theta_{ET}}. \tag{D.2.7}$$

The first-order conditions of (D.2.3) with respect to $\hat{C}$ and $\hat{F}$ are, respectively,

$$\frac{\hat{C}^{-\gamma}}{\left((1-\eta)\hat{J}\right)^{\frac{1-\gamma}{1-\eta}-1}} - \hat{\phi}'\left(\hat{i}\right)\hat{J}_{\hat{K}} = 0 \quad \Rightarrow \quad \hat{C} = \left(\hat{\phi}'\left(\hat{i}\right)\hat{J}_{\hat{K}}\right)^{-\frac{1}{\gamma}}\left((1-\eta)\hat{J}\right)^{-\frac{1}{\gamma}\frac{\eta-\gamma}{1-\eta}} \tag{D.2.8}$$

and

$$\hat{J}_{\hat{K}}\left((1-\alpha)\hat{A}\left(\hat{E}\right)\hat{K}^{\alpha}\hat{F}^{-\alpha} - \hat{b}\right)\hat{\phi}'\left(\hat{i}\right) + \hat{J}_{\hat{E}}e^{-\hat{g}\hat{t}}\hat{\mu} = 0 \quad \Rightarrow$$

$$\hat{F} = \left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}, \tag{D.2.9}$$

where we have defined the SCC in non-dimensional terms as

$$\hat{P} \equiv \left(\frac{F_0}{g_0 K_0}\right)P = -\hat{\mu}\frac{\hat{J}_{\hat{E}}}{\hat{\phi}'(\hat{i})\hat{J}_{\hat{K}}}, \tag{D.2.10}$$

and use (D.2.9) to write the production function as

$$\hat{Y} = \hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{K}^{\alpha}\hat{F}^{1-\alpha} = \hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}}\hat{K}. \tag{D.2.11}$$

D.3 Derivation of zeroth-order solution

In non-dimensional terms, the truncated series solutions for the value function and the forward-looking control variables (5.3.5) is given by

$$\begin{aligned}
\hat{J}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) &= \hat{J}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right) + \epsilon\hat{J}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) + O(\epsilon^2),\\
\hat{F}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) &= \hat{F}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right) + \epsilon\hat{F}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) + O(\epsilon^2),\\
\hat{C}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) &= \hat{C}^{(0)}\left(\hat{K},\epsilon\hat{D}(\hat{E},\hat{\chi},\hat{\lambda})\right) + \epsilon\hat{C}^{(1)}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) + O(\epsilon^2).
\end{aligned} \tag{D.3.1}$$

At $O(1)$ the Hamilton-Jacobi-Bellman equation (D.2.3) can be written as

$$\begin{aligned}
&\frac{\left(\hat{\phi}'(\hat{i}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{-\frac{1-\gamma}{\gamma}\frac{\eta-\gamma}{1-\eta}} - \hat{\rho}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}}}{(1-\gamma)\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1-\gamma}{1-\eta}-1}} + \hat{J}^{(0)}_{\hat{t}} + \hat{J}^{(0)}_{\hat{K}}\hat{\phi}(\hat{i}^{(0)})\hat{K}\\
&+ \frac{1}{2}\hat{J}^{(0)}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}_K^2\hat{J}_{\hat{K}\hat{E}}\hat{K}\hat{E}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \hat{J}_{\hat{K}\hat{\chi}}\hat{K}\hat{\chi}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi + \hat{J}_{\hat{K}\hat{\lambda}}\hat{K}\hat{\lambda}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\\
&+ \hat{J}_{\hat{E}\hat{\chi}}\hat{E}\hat{\chi}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi + \hat{J}_{\hat{E}\hat{\lambda}}\hat{E}\hat{\lambda}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda + \hat{J}_{\hat{\chi}\hat{\lambda}}\hat{\chi}\hat{\lambda}\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda = 0,
\end{aligned} \tag{D.3.2}$$

where we have substituted for the forward-looking variables $\hat{C}$ and $\hat{F}$ at $O(1)$ from (D.2.8) and (D.2.9) and we have used

$$\underbrace{\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(1)} = \hat{\phi}(\hat{\imath}^{(0)})\hat{K}. \tag{D.3.3}$$

In (D.3.2) and (D.3.3), $\hat{\imath}^{(0)}$ is the (constant) optimally chosen investment rate. Equation (D.3.2) has a power-law solution of the form $\hat{J}^{(0)} = \psi_0\hat{K}^{1-\eta}$, and following some manipulation we obtain

$$\hat{J}^{(0)} = \psi_0\hat{K}^{1-\eta} \text{ with } \psi_0 = \frac{1}{1-\eta}\left(\hat{\phi}'(\hat{\imath}^{(0)})\right)^{-(1-\eta)}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{\imath}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right)^{-\gamma\frac{1-\eta}{1-\gamma}}. \tag{D.3.4}$$

From the first-order condition (D.2.8), we obtain

$$\hat{C}^{(0)} = \hat{c}^{(0)}\hat{K} \quad \text{with} \quad \hat{c}^{(0)} = \frac{1}{\hat{\phi}'(\hat{\imath}^{(0)})}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{\imath}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right), \tag{D.3.5}$$

where $q\left(\hat{\imath}\right) = 1/\hat{\phi}'\left(\hat{\imath}\right)$ denotes Tobin's q, the price of capital in consumption terms.[1]

We can thus write the value function (D.3.4) as

$$\hat{J}^{(0)} = \frac{1}{1-\eta}\left(\hat{\phi}'(\hat{\imath}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}}\hat{K}^{1-\eta}. \tag{D.3.6}$$

Substituting in for $\hat{F}$ from the optimality condition (D.2.9) and for $\hat{Y}$ from (D.2.11), we obtain from $\hat{I} = \hat{Y} - \hat{C} - \hat{b}\hat{F}$:

$$\hat{\imath}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{c}^{(0)} = \hat{r}^{(0)}_{\text{mpk}} + \hat{\delta} - \hat{q}^{(0)}\left(\hat{\rho}-(1-\gamma)\left(\hat{\phi}(\hat{\imath}^{(0)})-\frac{1}{2}\eta\hat{\sigma}_K^2\right)\right), \tag{D.3.7}$$

where

$$\hat{r}^{(0)}_{\text{mpk}} \equiv \hat{Y}_{\hat{K}}(\hat{P}=0) - \hat{\delta} = \alpha\hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{\frac{1}{\alpha}}\left((1-\alpha)/\hat{b}\right)^{\frac{1-\alpha}{\alpha}} - \hat{\delta} \tag{D.3.8}$$

denotes the marginal productivity of capital net of depreciation.[2] Equation (D.3.7) implicitly defines the optimally chosen investment rate $\hat{\imath}^{(0)}$. From (D.3.3), the leading-order endogenous growth rate of capital and hence of consumption is given by

$$\hat{g}^{(0)} = \underbrace{\frac{1}{\hat{K}}\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(1)} = \hat{\phi}(\hat{\imath}^{(0)}) \tag{D.3.9}$$

[1] The value of the capital stock is $q\hat{K}$, or dimensionally qK, where $q = 1/\hat{\phi}'(\hat{\imath}) = 1/\phi'(i)$ is already a fraction and is left unchanged by the scaling (cf. $\omega i = \hat{\omega}\hat{\imath}$).

[2] Dimensionally, we have $r^{(0)}_{\text{mpk}} = \hat{r}^{(0)}_{\text{mpk}}g_0$.

and hence $\hat{g}^{(0)} = \hat{\phi}\left(\hat{i}^{(0)}\right) = 1$. In equilibrium, the marginal propensity to consume $\hat{c}^{(0)}/\hat{q}^{(0)}$ equals the expected return on investment $\hat{r}^{(0)}$ minus the growth rate of the economy $\hat{g}^{(0)}$. In turn, the expected return on investment equals the sum of the risk-free rate $\hat{r}_{\mathrm{rf}}^{(0)}$ and the risk premium $\Delta\hat{r}^{(0)}$. Hence,

$$\hat{c}^{(0)}/\hat{q}^{(0)} = \hat{r}^{(0)} - \hat{g}^{(0)} = \hat{r}_{\mathrm{rf}}^{(0)} + \Delta\hat{r}^{(0)} - \hat{g}^{(0)} \tag{D.3.10}$$

and with a risk premium of $\Delta\hat{r}^{(0)} = \eta\hat{\sigma}_K^2$ in the absence of any climate risk at zeroth-order, the risk-free rate is:

$$\hat{r}_{\mathrm{rf}}^{(0)} = \hat{\rho} + \gamma\hat{g}^{(0)} - (1+\gamma)\eta\hat{\sigma}_K^2/2. \tag{D.3.11}$$

Although $\hat{J}_{\hat{E}}^{(0)}$ can be computed from (D.3.6), a consistent leading-order estimate of the optimal SCC also requires $\hat{J}_{\hat{E}}^{(1)}$ and thus the next order in the perturbation expansion, i.e.

$$\hat{P} = -\frac{\hat{\mu}\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}\right)}{\hat{\phi}'(\hat{i}^{(0)})\hat{J}_{\hat{K}}^{(0)}}. \tag{D.3.12}$$

D.4 Derivation of first-order solution

D.4.1 Solution to multi-variate Ornstein-Uhlenbeck process

We define $\hat{k} \equiv k \equiv \log(K/K_0)$, so the vector of all four states

$$d\hat{\mathbf{x}} = \left\{d\hat{k}, d\hat{E}, d\hat{\chi}, d\hat{\lambda}\right\}^T \tag{D.4.1}$$

can be described by one multi-variate Ornstein-Uhlenbeck process (5.2.9), which is given in non-dimensional terms by

$$d\hat{\mathbf{x}} = \hat{\boldsymbol{\alpha}} - \hat{\boldsymbol{\nu}} \circ (\hat{\mathbf{x}} - \hat{\boldsymbol{\mu}})\, d\hat{t} + \hat{\mathbf{S}} d\hat{\mathbf{W}}_{\hat{t}}. \tag{D.4.2}$$

The growth rate vector (5.2.10), relevant to the capital and atmospheric carbon stock processes only, is given in non-dimensional terms by

$$\begin{aligned}\hat{\boldsymbol{\alpha}} &= \left(\frac{1}{d\hat{t}}\frac{E_t\left[d\hat{K}\right]}{\hat{K}} - \frac{1}{2}\hat{\sigma}_K^2,\ \frac{1}{d\hat{t}}E_t\left[d\hat{E}\right],\ 0,\ 0\right)^T \\ &= \left(\hat{\phi}(\hat{i}) - \frac{1}{2}\hat{\sigma}_K^2,\ \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}\hat{t}},\ 0,\ 0\right)^T,\end{aligned} \tag{D.4.3}$$

the mean reversion rate vector by $\hat{\boldsymbol{\nu}} = \left(0, \hat{\varphi}, \hat{\nu}_\chi, \hat{\nu}_\lambda\right)^T$, the vector of means by $\hat{\boldsymbol{\mu}}^T = (0,0,1,1)^T$, and the covariance matrix $\hat{\mathbf{S}}\hat{\mathbf{S}}^T$ has the form

$$\frac{1}{d\hat{t}}E_t\left[d\hat{\mathbf{x}}d\hat{\mathbf{x}}^T\right] = \hat{\mathbf{S}}\hat{\mathbf{S}}^T = \begin{pmatrix} \hat{\sigma}_K^2 & \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \\ \rho_{KE}\hat{\sigma}_K\hat{\sigma}_E & \hat{\sigma}_E^2 & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda \\ \rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi & \rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi & \hat{\sigma}_\chi^2 & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \\ \rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda & \rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda & \rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda & \hat{\sigma}_\lambda^2 \end{pmatrix}. \tag{D.4.4}$$

We begin by integrating the multi-variate Ornstein-Uhlenbeck process (D.4.2), including only terms at zeroth order, so that the coefficients are constant and a closed-form solution is available. Specifically,

$$\hat{\boldsymbol{\alpha}}^{(0)} = \left(\hat{\phi}(\hat{\imath}^{(0)}) - \hat{\sigma}_K^2/2, \hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K}_0, 0, 0\right)^T, \tag{D.4.5}$$

where we have relied on the solution for $\hat{K}$ from the zeroth-order problem (cf. (D.3.9)). The slow dependence of productivity $\hat{A}$ on the states $\hat{E}$, $\hat{\chi}$ and $\hat{\lambda}$ can be neglected when integrating with respect to time, consistent with the multiple-scales nature of our perturbation expansion. For constant coefficients, (D.4.2) can be integrated to give:

$$\hat{\mathbf{x}}(t) = \hat{\boldsymbol{\mu}} + \hat{\boldsymbol{\alpha}} t + e^{\hat{\boldsymbol{\nu}}\hat{t}} \circ (\hat{\mathbf{x}}_0 - \hat{\boldsymbol{\mu}}) + \int_0^{\hat{t}} e^{\hat{\boldsymbol{\nu}}(\hat{u}-\hat{t})} \circ \hat{\mathbf{S}} d\mathbf{W}_{\hat{u}}. \tag{D.4.6}$$

The quantity $\hat{\mathbf{x}}(t)$ is normally distributed with covariance matrix $\hat{\boldsymbol{\Sigma}}(t)$:

$$\hat{\boldsymbol{\Sigma}}(t) = \int_0^{\hat{t}} \left(e^{\hat{\nu}(\hat{u}-\hat{t})} \circ \hat{\mathbf{S}}\right)\left(e^{\hat{\nu}(\hat{u}-\hat{t})} \circ \hat{\mathbf{S}}\right)^T d\hat{u} \tag{D.4.7}$$

$$= \begin{pmatrix} \hat{\sigma}_K^2 \hat{t} & \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\nu}_\chi}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\lambda\hat{t}}\right) \\ \frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{\varphi}}\left(1-e^{-\hat{\varphi}\hat{t}}\right) & \frac{\hat{\sigma}_E^2}{2\hat{\varphi}}\left(1-e^{-2\hat{\varphi}\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\hat{\nu}_\chi}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-(\hat{\varphi}+\hat{\nu}_\lambda)\hat{t}}\right) \\ \frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{\nu}_\chi}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{\varphi}+\hat{\nu}_\chi}\left(1-e^{-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\hat{t}}\right) & \frac{\hat{\sigma}_\chi^2}{2\hat{\nu}_\chi}\left(1-e^{-2\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\hat{t}}\right) \\ \frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{\nu}_\lambda}\left(1-e^{-\hat{\nu}_\chi\hat{t}}\right) & \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{\varphi}+\hat{\nu}_\lambda}\left(1-e^{-(\hat{\varphi}+\hat{\nu}_\lambda)\hat{t}}\right) & \frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\left(1-e^{-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\hat{t}}\right) & \frac{\hat{\sigma}_\lambda^2}{2\hat{\nu}_\lambda}\left(1-e^{-2\hat{\nu}_\lambda\hat{t}}\right) \end{pmatrix}.$$

D.4.2 Evolution equations for $\hat{K}$ and $\hat{E}$

We consider the expected evolution equations of the states $\hat{K}$ and $\hat{E}$ at $O(\epsilon)$ and $O(1)$, respectively. At this order, we have for the expected evolution of $\hat{K}$:

$$\underbrace{\frac{1}{d\hat{t}} E_t\left[d\hat{K}\right]}_{O(\epsilon)} = \hat{\phi}'(\hat{\imath}^{(0)})\epsilon\hat{I}^{(1)} = -\hat{\phi}'(\hat{\imath}^{(0)})\epsilon\hat{C}^{(1)} = \frac{\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\hat{\phi}''}{\hat{\phi}'(\hat{\imath}^{(0)})}} \hat{K}\left(\frac{\epsilon\hat{J}_{\hat{K}}^{(1)}}{\hat{J}_{\hat{K}}^{(0)}} + \frac{\eta-\gamma}{1-\eta}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right), \tag{D.4.8}$$

where the first identity makes use of the identity $\hat{\Phi} = \epsilon\hat{I}^{(1)} - \hat{\omega}\epsilon\hat{I}^{(1)}\hat{I}^{(0)}/\hat{K} = \epsilon\hat{I}^{(1)}\hat{\phi}'(\hat{\imath}^{(0)})$ at $O(\epsilon)$. We further note from $\hat{I} = \hat{Y} - \hat{b}\hat{F} - \hat{C}$ that $\hat{I}^{(1)} = -\hat{C}^{(1)}$, since production net of fossil fuel costs is unaffected by the SCC in our formulation:

$$\frac{\partial}{\partial\hat{P}}\left[\hat{Y} - \hat{b}\hat{F}\right]\Big|_{P=0} =$$
$$\frac{\partial}{\partial\hat{P}}\left[\hat{A}^{\frac{1}{\alpha}}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1-\alpha}{\alpha}}\hat{K} - \hat{b}\left(\frac{1-\alpha}{\hat{b}+\hat{P}\exp(-\hat{g}\hat{t})}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}\right]\Bigg|_{P=0} = 0. \tag{D.4.9}$$

The identity in (D.4.9) relies on the Cobb-Douglas nature of the production function. The third identity in (D.4.8) follows from a Taylor-series expansion of $\hat{C}$, given by (D.2.8), with respect to the small parameter ϵ (about $\epsilon = 0$):

$$\hat{c}^{(1)} = \hat{c}^{(0)} \left(-\frac{1}{\gamma} \frac{\hat{\phi}''}{\hat{\phi}'\left(\hat{\imath}^{(0)}\right)} \epsilon \hat{\imath}^{(1)} - \frac{1}{\gamma} \frac{\epsilon \hat{J}^{(1)}_{\hat{K}}}{\hat{J}^{(0)}_{\hat{K}}} - \frac{1}{\gamma} \frac{\eta - \gamma}{1 - \eta} \frac{\epsilon \hat{J}^{(1)}}{\hat{J}^{(0)}} \right). \tag{D.4.10}$$

Noting that $\hat{\imath}^{(1)} = -\hat{c}^{(1)}$, we can rearrange this linear equation to give

$$\hat{c}^{(1)} = \frac{\hat{c}^{(0)}}{1 - \frac{1}{\gamma}\frac{\hat{c}^{(0)}\hat{\phi}''}{\hat{\phi}'(\hat{\imath}^{(0)})}} \left(-\frac{1}{\gamma} \frac{\hat{J}^{(1)}_{\hat{K}}}{\hat{J}^{(0)}_{\hat{K}}} - \frac{1}{\gamma} \frac{\eta - \gamma}{1 - \eta} \frac{\hat{J}^{(1)}}{\hat{J}^{(0)}} \right), \tag{D.4.11}$$

which is used in the third identity in (D.4.8). For $\hat{E}$, we have at $O(1)$:

$$\underbrace{\frac{1}{d\hat{t}} E_t \left[d\hat{E} \right]}_{O(1)} = \hat{\mu} \left(\frac{1 - \alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K} e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E}. \tag{D.4.12}$$

D.4.3 The Hamilton-Jacobi-Bellman equation

Substituting for the forward-looking variables $\hat{C}$ from (D.2.9) and $\hat{F}$ from (D.2.9), the Hamilton-Jacobi-Bellman equation (D.2.3) becomes at $O(\epsilon)$:

$$\begin{aligned}
&\underbrace{\hat{f}^*(\hat{J})}_{O(\epsilon)} + \epsilon \hat{J}^{(1)}_{\hat{t}} + \epsilon \hat{J}^{(1)}_{\hat{K}} \hat{K} \hat{\phi}(\hat{\imath}^{(0)}) + \frac{\hat{\phi}'\left(\hat{\imath}^{(0)}\right) \hat{c}^{(0)}}{\gamma - \hat{c}^{(0)}\hat{\phi}''/\hat{\phi}'(\hat{\imath}^{(0)})} \left(\hat{K}\hat{J}^{(1)}_{\hat{K}} + (\eta - \gamma)\hat{J}^{(1)} \right) \\
&+ \left(\hat{J}^{(0)}_{\hat{E}} + \epsilon \hat{J}^{(1)}_{\hat{E}} \right) \left(\hat{\mu} \left(\frac{1-\alpha}{\hat{b}} \right)^{\frac{1}{\alpha}} \hat{A}^{\frac{1}{\alpha}} \hat{K} e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E} \right) + \left(\hat{J}^{(0)}_{\hat{\chi}} + \epsilon \hat{J}^{(1)}_{\hat{\chi}} \right) \hat{\nu}_{\chi}(1 - \hat{\chi}) \\
&+ \left(\hat{J}^{(0)}_{\hat{\lambda}} + \epsilon \hat{J}^{(1)}_{\hat{\lambda}} \right) \hat{\nu}_{\lambda}(1 - \hat{\lambda}) + \frac{1}{2} \epsilon \hat{J}^{(1)}_{\hat{K}\hat{K}} \hat{K}^2 \hat{\sigma}_K^2 + \frac{1}{2} \left(\hat{J}^{(0)}_{\hat{E}\hat{E}} + \epsilon \hat{J}^{(1)}_{\hat{E}\hat{E}} \right) \hat{E}^2 \hat{\sigma}_E^2 \\
&+ \frac{1}{2} \left(\hat{J}^{(0)}_{\hat{\chi}\hat{\chi}} + \epsilon \hat{J}^{(1)}_{\hat{\chi}\hat{\chi}} \right) \hat{\sigma}_{\chi}^2 + \frac{1}{2} \left(\hat{J}^{(0)}_{\hat{\lambda}\hat{\lambda}} + \epsilon \hat{J}^{(1)}_{\hat{\lambda}\hat{\lambda}} \right) \hat{\sigma}_{\lambda}^2 + \left(\hat{J}^{(0)}_{\hat{K}\hat{E}} + \epsilon \hat{J}^{(1)}_{\hat{K}\hat{E}} \right) \hat{K} \rho_{KE} \hat{\sigma}_K \hat{\sigma}_E \\
&+ \left(\hat{J}^{(0)}_{\hat{K}\hat{\chi}} + \epsilon \hat{J}^{(1)}_{\hat{K}\hat{\chi}} \right) \hat{K} \rho_{K\chi} \hat{\sigma}_K \hat{\sigma}_{\chi} + \left(\hat{J}^{(0)}_{\hat{K}\hat{\lambda}} + \epsilon \hat{J}^{(1)}_{\hat{K}\hat{\lambda}} \right) \hat{K} \rho_{K\lambda} \hat{\sigma}_K \hat{\sigma}_{\lambda} \\
&+ \left(\hat{J}^{(0)}_{\hat{E}\hat{\chi}} + \epsilon \hat{J}^{(1)}_{\hat{E}\hat{\chi}} \right) \rho_{E\chi} \hat{\sigma}_E \hat{\sigma}_{\chi} + \left(\hat{J}^{(0)}_{\hat{E}\hat{\lambda}} + \epsilon \hat{J}^{(1)}_{\hat{E}\hat{\lambda}} \right) \rho_{E\lambda} \hat{\sigma}_E \hat{\sigma}_{\lambda} + \left(\hat{J}^{(0)}_{\hat{\chi}\hat{\lambda}} + \epsilon \hat{J}^{(1)}_{\hat{\chi}\hat{\lambda}} \right) \rho_{\chi\lambda} \hat{\sigma}_{\chi} \hat{\sigma}_{\lambda} = 0,
\end{aligned} \tag{D.4.13}$$

where we have used the identity $\partial/\partial\hat{k} = \hat{K}\partial/\partial\hat{K}$ (chain rule), we substituted the evolution equations for $\hat{K}$ at subsequent orders ((D.3.3) and (D.4.8)) and $\hat{E}$ at zeroth-order (D.4.12), and defined $\hat{f}^*(J) \equiv \hat{f}\left(\hat{C}^*, \hat{J}\right)$ with $\hat{C}$ optimally chosen. From (5.2.1) and (D.2.8), $\hat{f}^*(J)$ is

$$\hat{f}^* = \frac{1}{1-\gamma} \left(\hat{\phi}'(\hat{\imath}) \hat{J}_{\hat{K}} \right)^{-\frac{1-\gamma}{\gamma}} \left((1-\eta)\hat{J} \right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}} - \frac{1-\eta}{1-\gamma} \hat{\rho} \hat{J}. \tag{D.4.14}$$

A Taylor-series expansion for $\hat{f}^*(J)$ in ϵ (about $\epsilon = 0$) gives

$$\underbrace{\hat{f}^*}_{O(\epsilon)} = \frac{\left(\hat{\phi}'(\hat{\imath}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\gamma}$$
$$\times\left(-\frac{\hat{\phi}''}{\hat{\phi}'(\hat{\imath}^{(0)})}\epsilon\hat{\imath}^{(1)} - \frac{\epsilon\hat{J}^{(1)}_{\hat{K}}}{\hat{J}^{(0)}_{\hat{K}}} + \frac{\gamma-\eta}{(1-\gamma)(1-\eta)}\frac{\epsilon\hat{J}^{(1)}}{\hat{J}^{(0)}}\right)$$
$$-\frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)}\frac{\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}}{\gamma}$$
$$\times\left(-\frac{\hat{\phi}''}{\hat{\phi}'\left(\hat{\imath}^{(0)}\right)}\frac{\hat{c}^{(0)}}{\gamma - \frac{\hat{c}^{(0)}\hat{\phi}''}{\hat{\phi}'(\hat{\imath}^{(0)})}}\left(\epsilon\hat{K}\hat{J}^{(1)}_{\hat{K}} + (\eta-\gamma)\,\epsilon\hat{J}^{(1)}\right) - \epsilon\hat{K}\hat{J}^{(1)}_{\hat{K}} + \frac{\gamma-\eta}{(1-\gamma)}\epsilon\hat{J}^{(1)}\right)$$
$$-\frac{1-\eta}{1-\gamma}\hat{\rho}\epsilon\hat{J}^{(1)}, \tag{D.4.15}$$

where we have substituted for $\hat{\imath}^{(1)} = -\hat{c}^{(1)}$ from (D.4.11) and used the identity:

$$\frac{\left(\hat{\phi}'(\hat{\imath}^{(0)})\hat{J}^{(0)}_{\hat{K}}\right)^{-\frac{1-\gamma}{\gamma}}\left((1-\eta)\hat{J}^{(0)}\right)^{\frac{1}{\gamma}\frac{\gamma-\eta}{1-\eta}}}{\hat{K}\hat{J}^{(0)}_{\hat{K}}} = \hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}. \tag{D.4.16}$$

Substituting from (D.4.14), two of the terms in (D.4.13) simplify to

$$\underbrace{\hat{f}^*}_{O(\epsilon)} + \hat{J}^{(0)}_{\hat{K}}\underbrace{\frac{1}{d\hat{t}}E_t\left[d\hat{K}\right]}_{O(\epsilon)} = -\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}(\eta-\gamma) + (1-\eta)\hat{\rho}\right]\epsilon\hat{J}^{(1)}. \tag{D.4.17}$$

Using (D.4.17), (D.4.13) can be rewritten as a forced equation:

$$-\frac{1}{1-\gamma}\left[\hat{\phi}'(\hat{\imath}^{(0)})\hat{c}^{(0)}(\eta-\gamma) + (1-\eta)\hat{\rho}\right]\epsilon\hat{J}^{(1)} + \epsilon\hat{J}^{(1)}_{\hat{t}} + \epsilon\hat{J}^{(1)}_{\hat{K}}\hat{K}\hat{\phi}(\hat{\imath}^{(0)})$$
$$\epsilon\hat{J}^{(1)}_{\hat{E}}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}} - \hat{\varphi}\hat{E}\right) + \epsilon\hat{J}^{(1)}_{\hat{\chi}}\hat{\nu}_{\chi}(1-\hat{\chi}) + \epsilon\hat{J}^{(1)}_{\hat{\lambda}}\hat{\nu}_{\lambda}\left(1-\hat{\lambda}\right)$$
$$+\frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{K}\hat{K}}\hat{K}^2\hat{\sigma}^2_K + +\frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{E}\hat{E}}\hat{E}^2\hat{\sigma}^2_E + \frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{\chi}\hat{\chi}}\hat{\sigma}^2_{\chi} + \frac{1}{2}\epsilon\hat{J}^{(1)}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}^2_{\lambda}$$
$$\epsilon\hat{J}^{(1)}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E + \epsilon\hat{J}^{(1)}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi} + \epsilon\hat{J}^{(1)}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}$$
$$+\epsilon\hat{J}^{(1)}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi} + \epsilon\hat{J}^{(1)}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda} + \epsilon\hat{J}^{(1)}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda} = -\hat{G}\left(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}\right), \tag{D.4.18}$$

where the forcing is defined as

$$\hat{G}(\hat{t},\hat{K},\hat{E},\hat{\chi},\hat{\lambda}) \equiv \hat{J}^{(0)}_{\hat{E}}\left(\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{K}e^{-\hat{g}^{(0)}\hat{t}}-\hat{\varphi}\hat{E}\right)+\hat{J}^{(0)}_{\hat{\chi}}\hat{\nu}_{\chi}(1-\hat{\chi})+\hat{J}^{(0)}_{\hat{\lambda}}\hat{\nu}_{\lambda}(1-\hat{\lambda})$$
$$+\frac{1}{2}\hat{J}^{(0)}_{\hat{E}\hat{E}}\hat{\sigma}^2_E+\frac{1}{2}\hat{J}^{(0)}_{\hat{\chi}\hat{\chi}}\hat{\sigma}^2_{\chi}+\frac{1}{2}\hat{J}^{(0)}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}^2_{\lambda}+\hat{J}^{(0)}_{\hat{K}\hat{E}}\hat{K}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E$$
$$+\hat{J}^{(0)}_{\hat{K}\hat{\chi}}\hat{K}\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_{\chi}+\hat{J}^{(0)}_{\hat{K}\hat{\lambda}}\hat{K}\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_{\lambda}+\hat{J}^{(0)}_{\hat{E}\hat{\chi}}\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_{\chi}$$
$$+\hat{J}^{(0)}_{\hat{E}\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_{\lambda}+\hat{J}^{(0)}_{\hat{\chi}\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}. \tag{D.4.19}$$

To obtain derivatives of the zeroth-order value function with respect to $\hat{E},\hat{\chi}$ and $\hat{\lambda}$, we first differentiate with respect to the marginal productivity of capital $\hat{r}^{(0)}_{\text{mpk}}$, which depends on these three variables (via the chain rule of differentiation). From (D.3.6), we obtain:

$$\frac{\partial\hat{J}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}=\hat{J}^{(0)}\left(-(1-\eta)\frac{\hat{\phi}''(\hat{i}^{(0)})}{\hat{\phi}'(\hat{i}^{(0)})}+\gamma\frac{1-\eta}{\hat{c}^{(0)}}\right)\frac{\partial\hat{i}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}. \tag{D.4.20}$$

Since the Investment rate is implicitly defined, we get from (D.3.7) by implicit differentiation:

$$\frac{\partial\hat{i}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}=\frac{1}{\gamma-\hat{c}^{(0)}\hat{\phi}''(\hat{i}^{(0)})/\hat{\phi}'(\hat{i}^{(0)})}. \tag{D.4.21}$$

Combining (D.4.20) and (D.4.21), we obtain

$$\frac{\partial\hat{J}^{(0)}}{\partial\hat{r}^{(0)}_{\text{mpk}}}=\hat{J}^{(0)}\frac{1-\eta}{\hat{c}^{(0)}}=\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}. \tag{D.4.22}$$

Using the chain rule of differentiation, we find the individual terms that contribute to the forcing (D.4.19):

$$\begin{aligned}\underbrace{\hat{J}^{(0)}_{\hat{E}}}_{O(\epsilon}&=\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}\frac{\partial\hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{E}}\quad\text{and}\\ \underbrace{\hat{J}^{(0)}_{\hat{E}\hat{E}}}_{O(\epsilon)}&=\left(\hat{\phi}'(\hat{i}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\hat{K}^{1-\eta}\frac{\partial^2\hat{r}^{(0)}_{\text{mpk}}}{\partial\hat{E}^2},\end{aligned} \tag{D.4.23}$$

and similarly for derivatives with respect to $\hat{\chi}$ and $\hat{\lambda}$, as well as cross-derivatives. From the zeroth-order solution

$$\hat{r}^{(0)}_{\text{mpk}}=\alpha\hat{A}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)^{1/\alpha}\left((1-\alpha)/\hat{b}\right)^{(1-\alpha)/\alpha}-\hat{\delta} \tag{D.4.24}$$

and the non-dimensional total factor productivity (D.2.5) we obtain

$$\begin{aligned}
\frac{\partial \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{E}} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^{*}\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\lambda}(\hat{\lambda}),\\
\frac{\partial^2 \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{E}^2} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^{*}\theta_{ET}\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}-1}\hat{\mathrm{X}}(\hat{\chi})\hat{\lambda}(\hat{\lambda}),
\end{aligned} \tag{D.4.25a}$$

$$\begin{aligned}
\frac{\partial \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{\chi}} &= -\epsilon \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\lambda}(\hat{\lambda}),\\
\frac{\partial^2 \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{\chi}^2} &= -\epsilon \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\hat{\lambda}(\hat{\lambda}),
\end{aligned} \tag{D.4.25b}$$

$$\begin{aligned}
\frac{\partial \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{\lambda}} &= -\epsilon \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\lambda}_{\hat{\lambda}}(\hat{\lambda}),\\
\frac{\partial^2 \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{\lambda}^2} &= -\epsilon \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\lambda}_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda}),
\end{aligned} \tag{D.4.25c}$$

$$\begin{aligned}
\frac{\partial^2 \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{E}\partial\hat{\chi}} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^{*}\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\lambda}(\hat{\lambda}),\\
\frac{\partial^2 \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{E}\partial\hat{\lambda}} &= -\epsilon \hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{A}^{*}\left(1+\theta_{ET}\right)\hat{E}^{\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),\\
\frac{\partial^2 \hat{r}^{(0)}_{\mathrm{mpk}}}{\partial \hat{\chi}\partial\hat{\lambda}} &= -\epsilon \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}} \hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}_{\hat{\chi}}(\hat{\chi})\hat{\lambda}_{\hat{\lambda}}\left(\hat{\lambda}\right),
\end{aligned} \tag{D.4.25d}$$

where have used the following short-hands $\hat{\mathrm{X}}(\hat{\chi}) \equiv (\hat{\chi})^{1+\theta_{\chi T}}$ and $\hat{\lambda}(\hat{\lambda}) \equiv \left(\hat{\lambda}\right)^{1+\theta_{\lambda}}$, so $\hat{D} = \epsilon\hat{E}^{1+\theta_{ET}}\hat{\mathrm{X}}(\hat{\chi})\hat{\lambda}(\hat{\lambda})$. Equations (D.4.23) and (D.4.25) can be substituted into (D.4.19):

$$\begin{aligned}\hat{G}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) = -\epsilon\hat{A}(\hat{E},\hat{\chi})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\hat{A}^{*}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{\imath}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}\\ \times\Bigg[\Bigg(-(1+\theta_{ET})\hat{\varphi}\hat{\mathrm{X}}\hat{\lambda}+\hat{\nu}_{\chi}(1-\hat{\chi})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\lambda}+\hat{\nu}_{\lambda}(1-\hat{\lambda})\hat{\mathrm{X}}\hat{\lambda}_{\hat{\lambda}}\\ +\frac{1}{2}\hat{\sigma}_{\chi}^{2}\hat{\mathrm{X}}_{\hat{\chi}\hat{\chi}}\hat{\lambda}+\frac{1}{2}\hat{\mathrm{X}}\hat{\lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^{2}+(1-\eta)\hat{\mathrm{X}}_{\hat{\chi}}\hat{\lambda}\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi}\\ +(1-\eta)\hat{\mathrm{X}}\hat{\lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda}+\hat{\mathrm{X}}_{\hat{\chi}}\hat{\lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Bigg)\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}}\\ +(1+\theta_{ET})\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{\mathrm{X}}\hat{\lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}}e^{-\hat{g}^{(0)}\hat{t}}\\ +\frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\hat{\sigma}_{E}^{2}\hat{\mathrm{X}}\hat{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}\\ +((1-\eta)(1+\theta_{ET})\hat{\mathrm{X}}\hat{\lambda}\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E}\\ +(1+\theta_{ET})\hat{\mathrm{X}}_{\hat{\chi}}\hat{\lambda}\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\\ +(1+\theta_{ET})\hat{\mathrm{X}}\hat{\lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}})\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}}\Bigg].\end{aligned} \tag{D.4.26}$$

Because we are ultimately interested in $\hat{J}^{(1)}_{\hat{E}}$ for the computation of the social cost of carbon, we first differentiate (D.4.18) with respect to $\hat{E}$ and seek a solution for $\hat{J}^{(1)}_{\hat{E}}$ of the form $\hat{J}^{(1)}_{\hat{E}} = \psi_1\left(1+\theta_{ET}\right)\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})$, which gives (from (D.4.18)):[3]

$$-\hat{r}_{\Omega}\hat{\Omega}+\frac{1}{d\hat{t}}E_t\left[d\hat{\Omega}\right] = -\hat{\gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}), \tag{D.4.27}$$

where we have introduced the effective discount rate

$$\hat{r}_{\Omega} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} + (1-\eta)\left(\hat{\phi}(\hat{\imath}^{(0)}) - \frac{1}{2}\eta\hat{\sigma}_K^2\right) + \hat{\varphi}, \tag{D.4.28}$$

and the coefficient

$$\psi_1 \equiv \hat{A}^{*}\hat{A}(\hat{E},\hat{\chi},\hat{\lambda})^{\frac{1}{\alpha}-1}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1-\alpha}{\alpha}}\left(\hat{c}^{(0)}\right)^{-\gamma\frac{1-\eta}{1-\gamma}-1}\left(\hat{\phi}'(\hat{\imath}^{(0)})\right)^{-\frac{1-\eta}{1-\gamma}}. \tag{D.4.29}$$

[3] Dimensionally, we have $\Omega = E_0^{\theta_{ET}}\bar{\chi}^{1+\theta_{\chi T}}\overline{\lambda}^{1+\theta_{\lambda}}K_0^{1-\eta}\hat{\Omega}$.

The scaled forcing is defined by[4]

$$\begin{aligned}\hat{\Gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t}) \equiv \Big(& (1+\theta_{ET})\hat{\varphi}\hat{X}\hat{\lambda} - \hat{\nu}_{\chi}(1-\hat{\chi})\hat{X}_{\hat{\chi}}\hat{\lambda} - \hat{\nu}_{\lambda}(1-\hat{\lambda})\hat{X}\hat{\lambda}_{\hat{\lambda}} - \frac{1}{2}\hat{\sigma}_{\chi}^{2}\hat{X}_{\hat{\chi}\hat{\chi}}\hat{\lambda} \\ & - \frac{1}{2}\hat{X}\hat{\lambda}_{\hat{\lambda}\hat{\lambda}}\hat{\sigma}_{\lambda}^{2} - (1-\eta)\hat{X}_{\hat{\chi}}\hat{\lambda}\rho_{K\chi}\hat{\sigma}_{K}\hat{\sigma}_{\chi} - (1-\eta)\hat{X}\hat{\lambda}_{\hat{\lambda}}\rho_{K\lambda}\hat{\sigma}_{K}\hat{\sigma}_{\lambda} \\ & - \hat{X}_{\hat{\chi}}\hat{\lambda}_{\hat{\lambda}}\rho_{\chi\lambda}\hat{\sigma}_{\chi}\hat{\sigma}_{\lambda}\Big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}} \\ & - \theta_{ET}\hat{\mu}\left(\frac{1-\alpha}{\hat{b}}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}\hat{X}\hat{\lambda}\hat{K}^{2-\eta}\hat{E}^{\theta_{ET}-1}e^{-\hat{g}^{(0)}\hat{t}} \\ & - \frac{1}{2}(\theta_{ET}-1)\theta_{ET}\hat{\sigma}_{E}^{2}\hat{X}\hat{\lambda}\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-2} \\ & - \theta_{ET}\big((1-\eta)\hat{X}\hat{\lambda}\rho_{KE}\hat{\sigma}_{K}\hat{\sigma}_{E} + \hat{X}_{\hat{\chi}}\hat{\lambda}\rho_{E\chi}\hat{\sigma}_{E}\hat{\sigma}_{\chi} \\ & + \hat{X}\hat{\lambda}_{\hat{\lambda}}\rho_{E\lambda}\hat{\sigma}_{E}\hat{\sigma}_{\lambda}\big)\hat{K}^{1-\eta}\hat{E}^{\theta_{ET}-1}.\end{aligned} \tag{D.4.30}$$

Equation (D.4.27) has the closed-form solution:

$$\hat{\Omega} = E_t\left[\int_{\hat{t}}^{\infty}\hat{\gamma}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{s})e^{-\hat{r}_{\Omega}(\hat{s}-\hat{t})}d\hat{s}\right]. \tag{D.4.31}$$

We can now compute the SCC according to $\hat{P} = -\hat{\mu}\left(\hat{J}_{\hat{E}}^{(0)} + \epsilon\hat{J}_{\hat{E}}^{(1)}\right)/\hat{\phi}'(\hat{\iota}^{(0)})\hat{J}_{\hat{K}}^{(0)}$:

$$\hat{P} = \frac{\hat{\mu}\,\hat{\Theta}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)\hat{Y}\big|_{\hat{P}=0}}{\hat{r}^{*}}\left(1 - \frac{\hat{\Omega}(\hat{K},\hat{E},\hat{\chi},\hat{\lambda},\hat{t})}{\hat{E}^{\theta_{ET}}\hat{X}(\hat{\chi})\hat{\lambda}(\hat{\lambda})\hat{K}^{1-\eta}}\right) \quad \text{with} \quad \hat{\Theta} \equiv \frac{\hat{D}_{\hat{E}}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}{1-\hat{D}\left(\hat{E},\hat{\chi},\hat{\lambda}\right)}, \tag{D.4.32}$$

where we introduced $\hat{r}^{*} \equiv \hat{r}^{(0)} - \hat{g}^{(0)}$. Dimensionally, (D.4.32) corresponds to Result 1.

D.5 Leading-order effects of uncertainty

Following the additional assumptions outlined at the start of Section 5.4, this appendix derives closed-form solutions for the optimal risk-adjusted SCC based on Result 1.

D.5.1 Carbon stock dynamics

The expected value of the carbon stock is governed by the differential equation (D.4.12) with solution

$$E_t\left[\hat{E}(\hat{s})\right] = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s}) + \hat{\mu}^{*}\hat{K}(\hat{t})\left[1-\exp(-\hat{\varphi}\Delta\hat{s})\right]/\hat{\varphi} = \hat{E}(\hat{t})\exp(-\hat{\varphi}\Delta\hat{s})\hat{e}(\hat{t}), \tag{D.5.1}$$

where $\hat{\mu}^{*} \equiv \hat{\mu}\left((1-\alpha)/\hat{b}\right)^{\frac{1}{\alpha}}\hat{A}^{\frac{1}{\alpha}}$, $\Delta\hat{s} \equiv \hat{s}-\hat{t}$ and $\hat{e}(\Delta\hat{s}) = 1+(\hat{\mu}^{*}\hat{K}(\hat{t})/\hat{E}(\hat{t}))(\exp(\hat{\varphi}\Delta\hat{s})-1)/\hat{\varphi}$. Dimensionally, we define μ^{*} so that $\mu F^{(0)} = \mu^{*}K$, where μ does not have

[4]Dimensionally, we have $\Gamma = E_0^{1+\theta_{ET}}\bar{\chi}^{1+\theta_{\chi T}}\overline{\lambda}^{1+\theta_{\lambda}}K_0^{1-\eta}g_0\hat{\gamma}$.

units and μ^* has units of TtC/\$year. We can then obtain $\mu^* = \mu \left(A \left(1-\alpha\right)/b\right)^{1/\alpha}$ or $\hat{\mu}^* = \left(K_0/g_0 E_0\right)\mu^*$.

D.5.2 Leading-order forcing

To identify leading-order terms only, we expand in $\Delta\hat{\chi} \equiv \hat{\chi} - \hat{\mu}_\chi$, $\Delta\hat{\lambda} \equiv \hat{\lambda} - \hat{\mu}_\lambda$ and $\Delta E \equiv E - E_t\left[E\right]$ with the corresponding covariance matrix given by (D.4.7) (assumption II). We begin by considering terms that only involve capital stock uncertainty, which can be evaluated without further approximation. The probability density function for time $\hat{s}$, but with the expectation operator evaluated at time $\hat{t}$, is

$$f_{\hat{k}} = \frac{1}{\sqrt{2\pi\hat{\sigma}_K^2(\hat{s}-\hat{t})}} \exp\left(-\frac{1}{2}\left(\frac{(\hat{k}-\hat{\alpha}_k\hat{s})^2}{\hat{\sigma}_K^2(\hat{s}-\hat{t})}\right)\right), \tag{D.5.2}$$

where $\hat{\alpha}_k = \hat{\phi}(\hat{\imath}^{(0)}) - \hat{\sigma}_K^2/2$. Combining with the discount factor in (D.4.31) and an additional factor accounting for the decay of the atmospheric carbon stock, we have without further approximation

$$\begin{aligned} E_t\left[\left(\hat{K}(\hat{s})\right)^{1-\eta}\right]\exp\left(-\left(\hat{r}_\Omega + \theta_{ET}\hat{\varphi}\right)\left(\hat{s}-\hat{t}\right)\right) &= \left(\hat{K}(\hat{t})\right)^{1-\eta}\exp(-\hat{r}^\star\Delta\hat{s}) \quad \text{and} \\ E_t\left[\left(\hat{K}(\hat{s})\right)^{2-\eta}\right]\exp\left(-(\hat{r}_\Omega + \hat{g}^{(0)} + (\theta_{ET}-1)\hat{\varphi})\Delta\hat{s}\right) &= \left(\hat{K}(\hat{t})\right)^{2-\eta}\exp(-\hat{r}^{\star\star}\Delta\hat{s}), \end{aligned} \tag{D.5.3}$$

where we have introduced the new short-hands $\hat{r}^\star \equiv \hat{r}^* + (1+\theta_{ET})\hat{\varphi} = \hat{r}^{(0)} - \hat{g}^{(0)} + (1+\theta_{ET})\hat{\varphi}$ and $\hat{r}^{\star\star} \equiv \hat{r}^{(0)} - \hat{g}^{(0)} - (1-\eta)\hat{\sigma}_K^2 + \theta_{ET}\hat{\varphi} = \hat{r}^\star - (1-\eta)\hat{\sigma}_K^2 - \hat{\varphi}$ and note the use of alternative star symbols to denote rates corrected for atmospheric carbon stock decay. To leading order, we have for the terms involving the carbon stock:

$$\begin{aligned} E_t\left[\hat{E}^{\theta_{ET}}\right] &= \left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}}\left[1 + \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right] + O(\hat{\Sigma}_E^4), \\ E_t\left[\hat{E}^{\theta_{ET}-1}\right] &= \left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-1}\left[1 + \frac{1}{2}\left(\theta_{ET}-1\right)\left(\theta_{ET}-2\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right] + O(\hat{\Sigma}_E^4), \\ E_t\left[\hat{E}^{\theta_{ET}-2}\right] &= \left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-2}\left[1 + \frac{1}{2}\left(\theta_{ET}-2\right)\left(\theta_{ET}-3\right)\left(\frac{\hat{\Sigma}_E}{E_t\left[\hat{E}(\hat{s})\right]}\right)^2\right] + O(\hat{\Sigma}_E^4), \end{aligned} \tag{D.5.4}$$

where we let the subscript on $\hat{\Sigma}^2$ denote the relevant elements of the covariance matrix $\hat{\Sigma}$ (D.4.7) and we have ignored any contributions to uncertainty from new emissions through their dependence on uncertain future GDP. The following terms make a contribution to the forcing (D.4.30): $\mathrm{X}\Lambda$, $\mathrm{X}_{\hat{\chi}}\Lambda$, $(\hat{\chi}-\hat{\mu}_\chi)\mathrm{X}_{\hat{\chi}}\Lambda$, $\mathrm{X}\Lambda_{\hat{\lambda}}$, $(\hat{\lambda}-\hat{\mu}_\lambda)\mathrm{X}\Lambda_\lambda$, $\mathrm{X}_{\hat{\chi}\hat{\chi}}\Lambda$ and $\mathrm{X}\Lambda_{\hat{\lambda}\hat{\lambda}}$. Keeping only those terms contributing at leading order, we have

$$E_t\left[\mathrm{X}(\hat{\chi})\right] = \hat{\mu}_\chi^{1+\theta_{\chi T}}\left[1 + \frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_\chi}{\hat{\mu}_\chi}\right)^2\right] + O(\hat{\Sigma}_\chi^4),$$
$$E_t\left[\mathrm{X}_{\hat{\chi}}(\hat{\chi})\right] = \hat{\mu}_\chi^{\theta_{\chi T}}\left[(\theta_{\chi T}+1) + \frac{1}{2}(\theta_{\chi T}+1)\theta_{\chi T}(\theta_{\chi T}-1)\left(\frac{\hat{\Sigma}_\chi}{\hat{\mu}_\chi}\right)^2\right] + O(\hat{\Sigma}_\chi^4), \tag{D.5.5a}$$

$$E_t\left[(\hat{\chi}-\hat{\mu}_\chi)\mathrm{X}_{\hat{\chi}}(\hat{\chi})\right] = \hat{\mu}_\chi^{1+\theta_{\chi T}}\left[(\theta_{\chi T}+1)\theta_{\chi T}\left(\frac{\hat{\Sigma}_\chi}{\hat{\mu}_\chi}\right)^2\right] + O(\hat{\Sigma}_\chi^4),$$
$$E_t\left[\mathrm{X}_{\hat{\chi}\hat{\chi}}(\hat{\chi})\right] = \hat{\mu}_\chi^{\theta_{\chi T}-1}\left[(\theta_{\chi T}+1)\theta_{\chi T}\right] + O(\hat{\Sigma}_\chi^2), \tag{D.5.5b}$$

$$E_t\left[\Lambda(\hat{\lambda})\right] = \hat{\mu}_\lambda^{1+\theta_\lambda}\left[1 + \frac{1}{2}(\theta_\lambda+1)\theta_\lambda\left(\frac{\hat{\Sigma}_\lambda}{\hat{\mu}_\lambda}\right)^2\right] + O(\hat{\Sigma}_\lambda^4),$$
$$E_t\left[\Lambda_{\hat{\lambda}}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{\theta_\lambda}\left[(\theta_\lambda+1) + \frac{1}{2}(\theta_\lambda+1)\theta_\lambda(\theta_\lambda-1)\left(\frac{\hat{\Sigma}_\lambda}{\hat{\mu}_\lambda}\right)^2\right] + O(\hat{\Sigma}_\lambda^4), \tag{D.5.6a}$$

$$E_t\left[(\hat{\lambda}-\hat{\mu}_\lambda)\Lambda_{\hat{\lambda}}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{1+\theta_\lambda}\left[(\theta_\lambda+1)\theta_\lambda\left(\frac{\hat{\Sigma}_\lambda}{\hat{\mu}_\lambda}\right)^2\right] + O(\hat{\Sigma}_\lambda^4),$$
$$E_t\left[\Lambda_{\hat{\lambda}\hat{\lambda}}(\hat{\lambda})\right] = \hat{\mu}_\lambda^{\theta_\lambda-1}\left[(\theta_\lambda+1)\theta_\lambda\right] + O(\hat{\Sigma}_\lambda^2), \tag{D.5.6b}$$

where all the terms have been evaluated be at their equilibrium values, namely $\hat{\chi} = 1$ and $\hat{\lambda} = 1$, so that $\mathrm{X} = 1$ and, an assumption that will be made henceforth (assumption I). Using (D.5.3)-(D.5.6), we now consider the terms in the forcing (D.4.30) consecutively and let the subscript indices correspond to the sequence of terms in (D.4.30) (left to right). To consider the covariance terms in the forcing (D.4.30), we also expand in $\Delta\hat{k} \equiv \hat{k} - \left(\hat{\phi}(\hat{\iota}^{(0)}) - \hat{\sigma}_K^2/2\right)\hat{t}$ and only consider deviations from the zeroth-order mean consistent with our search for leading-order terms only. Considering the forcing terms in (D.4.30) consecutively, the following terms arise:

$$
\begin{aligned}
E_t\left[\Gamma_1\right] = (1+\theta_{ET})\hat{\varphi}\Bigg[& 1 + \frac{1}{2}\theta_{ET}\left(1+\theta_{ET}\right)\frac{\hat{\sigma}_E^2}{\left(E_t\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}} \\
& + \frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\hat{\sigma}_\chi^2\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi} + \frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\hat{\sigma}_\lambda^2\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda} \\
& + (1-\eta)\left(1+\theta_{ET}\right)\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}} \\
& + (1-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} \\
& + (1-\eta)\left(1+\theta_\lambda\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda} \\
& + \left(1+\theta_{ET}\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi} \\
& + \left(1+\theta_{ET}\right)\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda} \\
& + \left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg] E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{1+\theta_{ET}}
\end{aligned}
\tag{D.5.7}
$$

and thus

$$
\begin{aligned}
\frac{\hat{\Omega}_1}{\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}}} = (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\Bigg(& 1 + (1+\theta_{ET})\hat{\mu}^*\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^*} - (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\Bigg) \\
& \times (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\frac{1}{2}\theta_{ET}\left(\theta_{ET}+1\right)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^*} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\frac{\hat{\sigma}_\chi^2}{\hat{r}^*+2\hat{\nu}_\chi} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\frac{1}{2}\theta_\lambda\left(\theta_\lambda+1\right)\frac{\hat{\sigma}_\lambda^2}{\hat{r}^*+2\hat{\nu}_\lambda} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}(1-\eta)\left(\theta_{ET}+1\right)\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{\hat{E}}\frac{1}{\hat{r}^*} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}(1-\eta)\left(1+\theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi}{\hat{r}^*+\hat{\nu}_\chi} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}(1-\eta)\left(1+\theta_\lambda\right)\frac{\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda}{\hat{r}^*+\hat{\nu}_\lambda} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(\theta_{ET}+1\right)\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}}\frac{1}{\hat{r}^*+\hat{\nu}_\chi} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(\theta_{ET}+1\right)\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{E}}\frac{1}{\hat{r}^*+\hat{\nu}_\lambda} \\
& + (1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^*}\left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{r}^*+\hat{\nu}_\chi+\hat{\nu}_\lambda},
\end{aligned}
\tag{D.5.8}
$$

$$E_t\left[\Gamma_2\right] = \left(1+\theta_{\chi T}\right)\hat{v}_\chi \left(\theta_{\chi T}\hat{\sigma}_\chi^2 \frac{1-\exp\left(-2\hat{v}_\chi \Delta\hat{s}\right)}{2\hat{v}_\chi}\ (1-\eta)\,\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi \frac{1-\exp\left(-\hat{v}_\chi\Delta\hat{s}\right)}{\hat{v}_\chi}\right.$$
$$+ (1+\theta_{ET})\,\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\chi}$$
$$\left.+\ (1+\theta_\lambda)\,\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \frac{1-\exp\left(-\left(\hat{v}_\chi+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{v}_\chi+\hat{v}_\lambda}\right) E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{1+\theta_{ET}}, \tag{D.5.9}$$

and thus

$$\frac{\hat{\Omega}_2}{\hat{K}^{1-\eta}\hat{E}^{1+\theta_{ET}}} =$$
$$\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\hat{\sigma}_\chi^2 \left(\frac{1}{\hat{r}^*} - \frac{1}{\hat{r}^*+2\hat{v}_\chi} + (1+\theta_{ET})\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}} - \hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}} - \frac{1}{\left(\hat{r}^*+2\hat{v}_\chi\right)^2}\right)\right)$$
$$+\left(1+\theta_{\chi T}\right)(1-\eta)\,\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi \left(\frac{1}{\hat{r}^*} - \frac{1}{\hat{r}^*+\hat{v}_\chi} + (1+\theta_{ET})\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}} - \hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}} - \frac{1}{\left(\hat{r}^*+\hat{v}_\chi\right)^2}\right)\right)$$
$$+ (1+\theta_{ET})\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}}$$
$$\times\left(\frac{1}{\hat{r}^*} - \frac{1}{\hat{r}^*+\hat{v}_\chi} + \theta_{ET}\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}} - \hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}} - \frac{1}{\left(\hat{r}^*+\hat{v}_\chi\right)^2}\right) - \frac{\hat{v}_\chi}{\left(\hat{r}^*+\hat{v}_\chi\right)^2}\frac{\hat{\varphi}}{\hat{r}^*}\right)$$
$$+\left(1+\theta_{\chi T}\right)(1+\theta_\lambda)\,\frac{\hat{v}_\chi\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{v}_\chi+\hat{v}_\lambda}$$
$$\times\left(\frac{1}{\hat{r}^*} - \frac{1}{\hat{r}^*+\hat{v}_\chi+\hat{v}_\lambda} + (1+\theta_{ET})\left(\hat{\mu}^*\frac{\hat{K}}{\hat{E}} - \hat{\varphi}\right)\left(\frac{1}{\hat{r}^{*2}} - \frac{1}{\left(\hat{r}^*+\hat{v}_\chi+\hat{v}_\lambda\right)^2}\right)\right), \tag{D.5.10}$$

where we have expanded, for example, $1/\left(\hat{r}^*+\theta_{ET}\hat{\varphi}\right) = \left(1/\hat{r}^*\right)\left(1-\left(\theta_{ET}\hat{\varphi}/\hat{r}^*\right)\right) + O\left(\hat{\varphi}^2\right)$ for consistency and continue to do so. The subsequent terms in (D.4.30) give to leading order

$$E_t\left[\Gamma_3\right] = (1+\theta_\lambda)\,\hat{v}_\lambda \left(\theta_\lambda\hat{\sigma}_\lambda^2 \frac{1-\exp\left(-2\hat{v}_\lambda\Delta\hat{s}\right)}{2\hat{v}_\lambda}\ (1-\eta)\,\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \frac{1-\exp\left(-\hat{v}_\lambda\Delta\hat{s}\right)}{\hat{v}_\lambda}\right.$$
$$+ (1+\theta_{ET})\,\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{v}_\lambda}$$
$$\left.\times\left(1+\theta_{\chi T}\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \frac{1-\exp\left(-\left(\hat{v}_\chi+\hat{v}_\lambda\right)\Delta\hat{s}\right)}{\hat{v}_\chi+\hat{v}_\lambda}\right) E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{1+\theta_{ET}}, \tag{D.5.11}$$

and thus

$$
\begin{aligned}
E_t\left[\Gamma_1\right] = (1+\theta_{ET})\hat{\varphi}\Bigg[& 1 + \frac{1}{2}\theta_{ET}\left(\theta_{ET}-1\right)\frac{\hat{\sigma}_E^2}{\left(E_t\left[\hat{E}\right]\right)^2}\frac{1-\exp\left(-2\hat{\varphi}\Delta\hat{s}\right)}{2\hat{\varphi}} \\
& + \frac{1}{2}\theta_{\chi T}\left(1+\theta_{\chi T}\right)\hat{\sigma}_\chi^2\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi} + \frac{1}{2}\theta_\lambda\left(1+\theta_\lambda\right)\hat{\sigma}_\lambda^2\frac{1-\exp\left(-2\hat{\nu}_\lambda\Delta\hat{s}\right)}{2\hat{\nu}_\lambda} \\
& + (1-\eta)\,\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\hat{\varphi}\Delta\hat{s}\right)}{\hat{\varphi}} \\
& + (1-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} \\
& + (1-\eta)\left(1+\theta_\lambda\right)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\frac{1-\exp\left(-\hat{\nu}_\lambda\Delta\hat{s}\right)}{\hat{\nu}_\lambda} \\
& + \theta_{ET}\left(1+\theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi} \\
& + \theta_{ET}\left(1+\theta_\lambda\right)\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda} \\
& + \left(1+\theta_{\chi T}\right)\left(1+\theta_\lambda\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg] E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},
\end{aligned}
\tag{D.5.12}
$$

and

$$
\begin{aligned}
E_t\left[\Gamma_2\right] = \hat{\nu}_\chi(1+\theta_{\chi T})\Bigg(& \theta_{\chi T}\hat{\sigma}_\chi^2\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi}\;(1-\eta)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} \\
& + \theta_{ET}\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{E_t\left[\hat{E}\right]}\frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\chi\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\chi} \\
& + \left(1+\theta_\lambda\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Bigg) E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},
\end{aligned}
\tag{D.5.13}
$$

$$
\begin{aligned}
E_t[\Gamma_3] = {} & \hat{\nu}_\lambda (1+\theta_\lambda) \Bigg(\theta_\lambda \hat{\sigma}_\lambda^2 \frac{1-\exp(-2\hat{\nu}_\lambda \Delta\hat{s})}{2\hat{\nu}_\lambda} \; (1-\eta)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda \frac{1-\exp(-\hat{\nu}_\lambda\Delta\hat{s})}{\hat{\nu}_\lambda} \\
& + \theta_{ET} \frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{E_t\left[\hat{E}\right]} \frac{1-\exp\left(-\left(\hat{\varphi}+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\varphi}+\hat{\nu}_\lambda} \\
& \times \left(1+\theta_{\chi T}\right)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda \frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda} \Bigg) \qquad \text{(D.5.14)} \\
& \times E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}},
\end{aligned}
$$

$$
E_t[\Gamma_4] = -\frac{1}{2}\left(1+\theta_{\chi T}\right)\theta_{\chi T}\hat{\sigma}_\chi^2 E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}, \tag{D.5.15}
$$

$$
E_t[\Gamma_5] = -\frac{1}{2}(1+\theta_\lambda)\theta_\lambda\hat{\sigma}_\lambda^2 E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}, \tag{D.5.16}
$$

$$
E_t[\Gamma_6] = -(1-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}, \tag{D.5.17}
$$

$$
E_t[\Gamma_7] = -(1-\eta)(1+\theta_\lambda)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}, \tag{D.5.18}
$$

$$
E_t[\Gamma_8] = -\left(1+\theta_{\chi T}\right)(1+\theta_\lambda)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_{\lambda t} E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}}, \tag{D.5.19}
$$

$$
\begin{aligned}
E_t[\Gamma_9] = {} & -\theta_{ET}\hat{\mu}^*\Big[1+\frac{1}{2}\theta_{\chi T}\left(\theta_{\chi T}+1\right)\hat{\sigma}_\chi^2\frac{1-\exp\left(-2\hat{\nu}_\chi\Delta\hat{s}\right)}{2\hat{\nu}_\chi} \qquad \text{(D.5.20)} \\
& + \frac{1}{2}\theta_\lambda(\theta_\lambda+1)\hat{\sigma}_\lambda^2\frac{1-\exp(-2\hat{\nu}_\lambda\Delta\hat{s})}{2\hat{\nu}_\lambda} \\
& + (2-\eta)\left(1+\theta_{\chi T}\right)\rho_{K\chi}\hat{\sigma}_K\hat{\sigma}_\chi\frac{1-\exp\left(-\hat{\nu}_\chi\Delta\hat{s}\right)}{\hat{\nu}_\chi} \\
& + (2-\eta)(1+\theta_\lambda)\rho_{K\lambda}\hat{\sigma}_K\hat{\sigma}_\lambda\frac{1-\exp(-\hat{\nu}_\lambda\Delta\hat{s})}{\hat{\nu}_\lambda} \qquad \text{(D.5.21)} \\
& + \left(1+\theta_{\chi T}\right)(1+\theta_\lambda)\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda\frac{1-\exp\left(-\left(\hat{\nu}_\chi+\hat{\nu}_\lambda\right)\Delta\hat{s}\right)}{\hat{\nu}_\chi+\hat{\nu}_\lambda}\Big] \\
& \times E_t\left[\hat{K}^{2-\eta}\right]e^{-\hat{g}^{(0)}\Delta\hat{s}}\left(E_t\left[\hat{E}(\hat{s})\right]\right)^{\theta_{ET}-1},
\end{aligned}
$$

$$
E_t[\Gamma_{10}] = -\frac{1}{2}\theta_{ET}(\theta_{ET}-1)\hat{\sigma}_E^2 E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-2}, \tag{D.5.22}
$$

$$
E_t[\Gamma_{11}] = -(1-\eta)\theta_{ET}\rho_{KE}\hat{\sigma}_K\hat{\sigma}_E E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1}, \tag{D.5.23}
$$

$$
E_t[\Gamma_{12}] = -(1+\theta_{\chi T})\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1}, \tag{D.5.24}
$$

$$
E_t[\Gamma_{13}] = -\theta_{ET}(1+\theta_\lambda)\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda E_t\left[\hat{K}^{1-\eta}\right]\left(E_t\left[\hat{E}\right]\right)^{\theta_{ET}-1}, \tag{D.5.25}
$$

where elements of the covariance matrix have been substituted from (D.4.7).

D.5.3 Leading-order solution

Combining all the leading-order terms in the forcing equations (D.5.12)-(D.5.25) and substituting into (D.4.32), we obtain after considerable manipulation

$$\hat{P} = \frac{\hat{\mu}\ \hat{\Theta}\left(\hat{E}, \hat{\chi}, \hat{\lambda}\right) \hat{Y}\big|_{\hat{P}=0}}{\hat{r}^{\star}} \left(1 + \theta_{ET}\hat{\mu}^{*}\frac{\hat{K}}{\hat{E}}\frac{1}{\hat{r}^{\star\star}}\Upsilon + \Delta_{EE} + \Delta_{\chi\chi} + \Delta_{\lambda\lambda} + \Delta_{CK}\right), \tag{D.5.26}$$

where

$$\begin{aligned}
\Delta_{EE} &\equiv \frac{1}{2}\theta_{ET}\left(\theta_{ET} - 1\right)\frac{\hat{\sigma}_E^2}{\hat{E}^2}\frac{1}{\hat{r}^{\star} - 2\hat{\varphi}}\Upsilon_{EE},\\
\Delta_{\chi\chi} &\equiv \frac{1}{2}\left(1 + \theta_{\chi T}\right)\theta_{\chi T}\frac{\hat{\sigma}_\chi^2}{\hat{r}^{\star} + 2\hat{v}_\chi}\Upsilon_{\chi\chi},\\
\Delta_{\lambda\lambda} &\equiv \frac{1}{2}\theta_{\lambda}\left(1 + \theta_{\lambda}\right)\frac{\hat{\sigma}_\lambda^2}{\hat{r}^{\star} + 2\hat{v}_\lambda}\Upsilon_{\lambda\lambda},\\
\Delta_{CK} &\equiv -(\eta - 1)\hat{\sigma}_K\\
&\qquad \times\left(\theta_{ET}\frac{\rho_{KE}\hat{\sigma}_E}{\hat{E}\left(\hat{r}^{\star} - \hat{\varphi}\right)}\Upsilon_{KE} + \left(1 + \theta_{\chi T}\right)\frac{\rho_{K\chi}\hat{\sigma}_\chi}{\hat{r}^{\star} + \hat{v}_\chi}\Upsilon_{K\chi} + \left(1 + \theta_{\lambda}\right)\frac{\rho_{K\lambda}\hat{\sigma}_\lambda}{\hat{r}^{\star} + \hat{v}_\lambda}\Upsilon_{K\lambda}\right),\\
\Delta_{CC} &\equiv \theta_{ET}\left(1 + \theta_{\chi T}\right)\frac{\rho_{E\chi}\hat{\sigma}_E\hat{\sigma}_\chi}{\hat{E}}\frac{\hat{r}^{\star}}{\hat{r}^{\star} - \hat{\varphi}}\frac{1}{\hat{r}^{\star} + \hat{v}_\chi}\Upsilon_{E\chi} + \left(1 + \theta_{\lambda}\right)\\
&\qquad \times\left(\left(1 + \theta_{\chi T}\right)\frac{\rho_{\chi\lambda}\hat{\sigma}_\chi\hat{\sigma}_\lambda}{\hat{r}^{\star} + \hat{v}_\lambda + \hat{v}_\lambda}\Upsilon_{\chi\lambda} + \theta_{ET}\frac{\rho_{E\lambda}\hat{\sigma}_E\hat{\sigma}_\lambda}{\hat{E}}\frac{\hat{r}^{\star}}{\hat{r}^{\star} - \hat{\varphi}}\frac{1}{\hat{r}^{\star} + \hat{v}_\lambda}\Upsilon_{E\lambda}\right),
\end{aligned} \tag{D.5.27}$$

and the correction factors are given by

$$
\begin{aligned}
\Upsilon &= \frac{\hat{r}^{\star\star}}{1-(1+\theta_{ET})\hat{\varphi}/\hat{r}^{\star}} \\
&\quad \int_0^\infty \left(\exp(-\hat{r}^{\star\star}\Delta\hat{s}) - \frac{(1+\theta_{ET})\hat{\varphi}}{\hat{r}^{\star}}\exp(-(\hat{r}^{\star}-\hat{\varphi})\Delta\hat{s})\right)(\hat{e}(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s}, \\
\Upsilon_{Ki} &= 1+\theta_{ET}\hat{\mu}^{*}\frac{\hat{K}}{\hat{E}}\frac{1}{1-(1+\theta_{ET})\hat{\varphi}/\hat{r}^{\star}}\Bigg[\left(1+\frac{(1+\theta_{ET})\hat{\varphi}}{\hat{\nu}_i}\right) \\
&\quad \int_0^\infty \exp\left(-(\hat{\nu}_i+\hat{r}^{\star}-\hat{\varphi})\Delta\hat{s}\right)(\hat{e}(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s} \\
&\quad -(1+\theta_{ET})\frac{\hat{\varphi}}{\hat{r}^{\star}}\frac{\hat{r}^{\star}+\hat{\nu}_i}{\hat{\nu}_i}\int_0^\infty \exp(-(\hat{r}^{\star}-\hat{\varphi})\Delta\hat{s})\,(\hat{e}(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s} \\
&\quad +\frac{2-\eta}{1-\eta}\frac{\hat{\nu}_i+\hat{r}^{\star}}{\hat{\nu}_i}\int_0^\infty \left(\exp(-\hat{r}^{\star\star}\Delta\hat{s})-\exp(-(\hat{r}^{\star\star}+\hat{\nu}_i)\Delta\hat{s})\right)(\hat{e}(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s}\Bigg] \\
&\quad \text{for } i=\chi,\lambda, \\
\Upsilon_{ij} &= 1+\theta_{ET}\hat{\mu}^{*}\frac{\hat{K}}{\hat{E}}\frac{1}{1-(1+\theta_{ET})\hat{\varphi}/\hat{r}^{\star}}\Bigg[\left(1+\frac{(1+\theta_{ET})\hat{\varphi}}{\hat{\nu}_i+\hat{\nu}_j}\right) \\
&\quad \int_0^\infty \exp\left(-(\hat{r}^{\star}+\hat{\nu}_i+\hat{\nu}_j-\hat{\varphi})\Delta\hat{s}\right)(\hat{e}\,(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s} \\
&\quad -\frac{(1+\theta_{ET})\hat{\varphi}}{\hat{\nu}_i+\hat{\nu}_j}\frac{\hat{r}^{\star}+\hat{\nu}_i+\hat{\nu}_j}{\hat{r}^{\star}}\int_0^\infty \exp(-(\hat{r}^{\star}-\hat{\varphi})\Delta\hat{s})\,(\hat{e}\,(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s} \\
&\quad +\frac{\hat{r}^{\star}+\hat{\nu}_i+\hat{\nu}_j}{\hat{\nu}_i+\hat{\nu}_j} \\
&\quad \int_0^\infty \left(\exp(-\hat{r}^{\star\star}\Delta\hat{s})-\exp(-(\hat{r}^{\star\star}+\hat{\nu}_i+\hat{\nu}_j)\Delta\hat{s})\right)(\hat{e}\,(\Delta\hat{s}))^{\theta_{ET}-1}\, d\Delta\hat{s}\Bigg] \\
&\quad \text{for } i,j=\chi,\lambda,
\end{aligned}
\tag{D.5.28}
$$

and we do not explicitly give the correction factors for the terms involving carbon stock uncertainty, as these terms are negligibly small.

Dimensionally, (D.5.26) together with (D.5.27) and (D.5.28) gives, using the definitions summarized in (D.1.1) and (D.1.5) and $\hat{\mu}^*\hat{K} = \mu F^{(0)}/(g_0 E_0))$, Result 2′ stated in Appendix D.1. This generalizes Result 2 to convex reduced-form damages, non-zero carbon stock volatility and potentially skewed damage uncertainty.

D.6 Calibration

D.6.1 Asset returns, risk aversion and intertemporal substitution

To calibrate the non-climatic part of our model to match historical asset returns, we follow the calibration of Pindyck and Wang (2013), but ignore the effect of catastrophic shocks considered by these authors.[5,6] Using monthly asset data from the S&P 500 for the period 1947-2008, we obtain an annual return on assets (capital gains plus dividends) of $r^{(0)} = 7.2\%$/year with annual volatility of $\sigma_K = 12\%$. For a return on safe assets of 0.80%/year based on the annualized monthly return on 3-months T-bills, we obtain a risk premium of $\Delta r^{(0)} \equiv r^{(0)} - r_{\mathrm{rf}}^{(0)} = 6.4\%$/year and calibrate the coefficient of relative risk aversion as $\eta = 4.3$ (cf. $\Delta r^{(0)} = \eta\sigma_K^2$). Taking the growth rate to be equal to the historical growth rate of $g^{(0)} = 2.0\%$/year, the equation $r_{\mathrm{rf}}^{(0)} = \rho + \gamma g^{(0)} - (1+\gamma)\eta\sigma_K^2/2$ (cf. (D.2.11)) defines the combinations of ρ and γ that are consistent with historical asset returns. Setting the coefficient of elasticity of intertemporal substitution $EIS = 2/3$, we obtain $\gamma = EIS^{-1} = 1.5$ and thus a rate of time preference is $\rho = 5.8\%$/year.

D.6.2 Productivity, fossil fuel, adjustment costs and the depreciation rate

To calibrate total factor productivity, we consider the production function in the absence of climate damage that can be obtained by setting $P = 0$ (*i.e.* at zeroth order), namely $Y^{(0)} = A^*K$ with $A^* = A^{1/\alpha}\left((1-\alpha)/b\right)^{(1-\alpha)/\alpha}$ (cf. (D.2.11)). Pindyck and Wang (2013) use empirical estimates of the physical, human and intangible capital stocks and find $A^* = 0.113$/year, which we adopt. Based on emissions of $F_0^{(0)} = 9.1$ GtC/year in 2015, energy costs making up a share $1 - \alpha = 6.6\%$ of world GDP at \$75 trillion/year, we obtain an estimate of the cost of fossil fuel of $b = Y_0^{(0)}(1-\alpha)/F_0^{(0)} = \$5.4 \times 10^2$/tC.[7] From this, we estimate the gross marginal productivity of capital $\left.Y_K^{(0)}\right|_{t=0} = \alpha A^* = 0.106$/year.[8] Us-

[5]Pindyck and Wang (2013) use Poisson shocks to capture small risks of large disasters (cf. Barro 2006) and thus match skewness and kurtosis of asset returns. These shocks are responsible for approximately 1%-point of the risk premium. We furthermore calibrate the zeroth-order or non-climatic part of our model based on historical GDP data, which may have included the (small) effects of climate in reality.

[6]The alternative is to calibrate our AK model to the observed volatility of consumption or output (cf. Gollier 2012), which are generally much less volatile than capital (asset returns). Because the volatilities of capital, consumption and output are equal to the volatility of capital in an AK model, this alternative calibration gives a much lower volatility and, consequently, a higher coefficient of relative risk aversion to match the equity premium (see also the discussion in Pindyck and Wang 2013). Historical data for the growth rate of world GDP for 1961-2015 imply an annual volatility of $\sigma_K = 1.5\%$ and thus a much higher value of risk aversion of $\eta = 2.8 \times 10^2$ for an equity premium of 6.4%/year. Kocherlota (1996) obtains $\sigma_K = 3.6\%$/year from US annual consumption growth during 1889-1978, which gives $\eta = 49$.

[7] We estimate the share of energy costs from data for energy use and energy costs from BP Statistical Review of World Energy 2017. Data for emissions are obtained from the same source available online at `https://www.bp.com/en/global/corporate/energy-economics/statistical-review-of-world-energy.html`. Our estimate of energy costs as a percentage of GDP is in good agreement with data from the U.S. Energy Information Administration available online at `https://www.eia.gov/totalenergy/data/annual/showtext.php?t=ptb0105`.

[8]This is in line with Caselli and Feyer (2007), who estimate annual marginal products of capital of 8.5%/year for rich countries and 6.9%/year for poor countries, and an observed annual risk pre-

Table D.1: Atmospheric carbon stock calibration.

Time	μ	φ [%/year]	$\sigma_{\mathbf{E}}$ [GtC/year$^{1/2}$]	$\sigma_{\mathbf{E}}/\mathbf{S_0}$ [%/year$^{1/2}$]	$\sigma_{\mathbf{E}}/\mathbf{E_0}$ [%/year$^{1/2}$]
1750-2004	1.0	0.66	0.31	0.036	0.12
1800-2004	0.75	0.00	0.26	0.029	0.10
1900-2004	0.59	0.00	0.21	0.025	0.081
1959-2004	0.79	0.91	0.23	0.027	0.089

ing Pindyck and Wang's (2013) consumption-investment ratio $c^{(0)}/i^{(0)} = 2.84$ and the identity $\alpha A^* = c^{(0)} + i^{(0)}$, we obtain initial values of $c^{(0)} = 7.84\%$/year and $i^{(0)} = 2.76\%$/year. Using $q^{(0)} = c^{(0)}/(r^{(0)} - g^{(0)}) = 1.51$ and $q^{(0)} = (1 - \omega i^{(0)})^{-1}$, we obtain for the adjustment-cost parameter $\omega = 12.2$ year. Finally, we find the depreciation rate that is consistent with the assumed rate of economic growth: $\delta = i^{(0)} - \omega(i^{(0)})^2/2 - g^{(0)} = 0.30\%$/year.

D.6.3 Atmospheric carbon stock and uncertainty

We calibrate our stylized carbon stock model (5.2.4) to the Law Dome Ice Core 2000-year data set and historical emissions. The first column of Figure D.1 shows maximum-likelihood estimates of the two parameters of our simple atmospheric carbon stock model (5.2.4) for different time periods, from which it is evident that estimates displaying a certain linear relationship between φ and μ are of comparable likelihood.[9] These loci of maximum likelihood are shown separately in Figure D.2, with the overall maximum denoted by a red circle and corresponding values given in Table D.1.

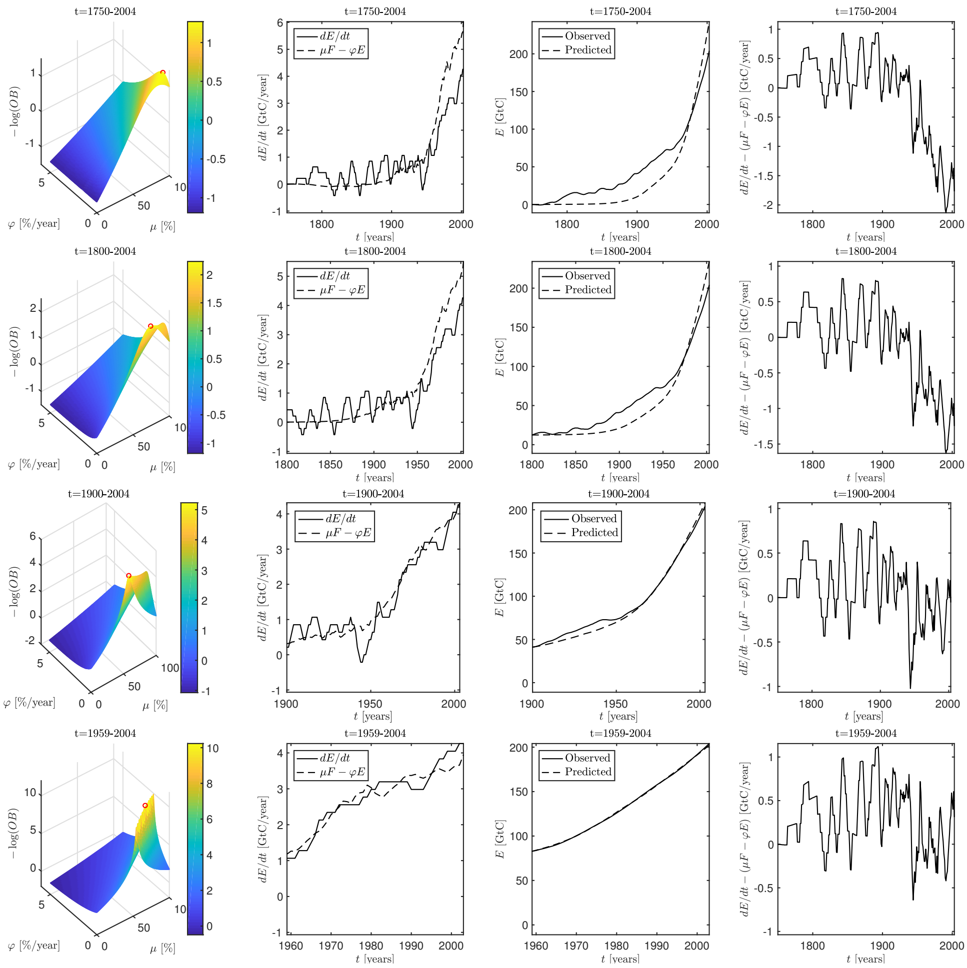
The remaining columns in Figure D.1 show the predicted and observed rate of change of the atmospheric carbon stock (second column), the predicted and observed atmospheric carbon stock (third column) and the remaining variability (fourth column).[10] For our base case, we set $\mu = 1.0$ and $\varphi = 0.66\%$/year. Fig-

mium of 5-7%/year. They use a depreciation rate of 6.0%/year to calculate the capital stock from investment, include the share of reproducible capital rather than the share of total capital, account for differences in prices between capital and consumption goods and correct for inflation.

[9]Annual data from the Law Dome firn and ice core records and the Cape Grim record are available online at `ftp://ftp.ncdc.noaa.gov/pub/data/paleo/icecore/antarctica/law/law2006.txt`. This data is based on spline fits to different dataset with different spline windows across time reflecting changes in the temporal resolution of the data. The discrete nature of the fitted data is evident for the early years. Annual carbon emissions from fossil fuel consumption and cement production are available online at `http://cdiac.ornl.gov/trends/emis/tre_glob_2013.html`.

[10]By setting $\varphi = 0$, we can estimate the fraction μ of emissions that stays in the atmosphere forever, whilst the remainder is instantaneously absorbed by the oceans and other carbon sinks. Calibrating to this data, we find $\mu = 0.68$, 0.64, 0.56 and 0.43 for the periods 1750-2004, 1800-2004, 1900-2004 and 1959-2004, respectively. Performing a similar analysis, Le Quéré et al. (2009) find that, between

Figure D.1: Atmospheric carbon stock calibration.

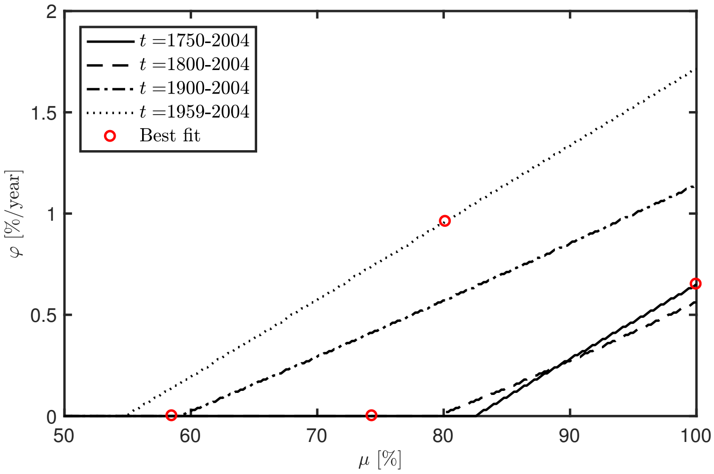


ure D.1 indicates that our simple model (5.2.4) captures the observed variation in the atmospheric carbon stock reasonably well, including for very long time periods. The final column in Table D.1 shows our estimates of the volatility as a percentage of the initial carbon stock, from which it is evident that the stochastic carbon stock correction to the optimal carbon price (D.5.27) will be extremely small.[11]

1959 and 2008, 43% of each year's CO_2 emissions remained in the atmosphere on average.

[11]We set the initial atmospheric carbon concentration at $t = 0$ to $S_0 = 401$ ppm of CO_2 (May 2015), corresponding to 0.854 TtC or 3.13 $TtCO_2$, and the pre-industrial atmospheric carbon concentration to 280 ppm CO_2, 0.596 TtC or 2.19 $TtCO_2$, so that $E_0 = 121$ ppm CO_2, 0.258 TtC or 0.94 $TtCO_2$. Updated and historical values can be found online at `http://www.esrl.noaa.gov/gmd/ccgg/trends/global.html`. We use the conversion factors: 1 ppm of CO_2 corresponds to 2.13 GtC and 1 GtC corresponds to 3.664 $GtCO_2$.

Figure D.2: Loci of best fit of the atmospheric carbon stock calibration.



D.6.4 Curvature of the temperature-carbon stock relationship

It is common to assume a logarithmic relationship between temperature and atmospheric carbon stock (Nordhaus 2008, Golosov et al. 2014, Hambel et al. 2017), thus introducing concavity. In our model, the normalized curvature of the temperature relationship (5.2.5) is constant: $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$. The radiative law for global mean temperature,

$$T \propto \ln(S/S_{PI})/\ln(2) \propto \ln\left((E + S_{PI})/S_{PI}\right)/\ln(2) \tag{D.6.1}$$

(Arrhenius 1896)[12] gives $\theta_E = -E/(E + S_{PI})$. If we evaluate (5.2.5) at double (quadruple) the pre-industrial stock $E = S_{PI}$ ($E = 3S_{PI}$), we obtain $\theta_E = -0.50$ ($\theta_E = -0.75$).[13] For $S_0 = 0.854$ TtC or $E_0 = 0.258$ TtC (given $S_{PI} = 0.596$ TtC), we obtain $\theta_E = -0.30$. Alternatively, using the simulations reported in Allen et al. (2009)[14] for the peak CO_2-induced warming as a function of cumulative emissions shown in Figure D.3, which we take to be equivalent to the transient climate response to cumulative emissions (TCRE), we estimate the curvature of our

[12]In their Table 6.2, IPCC (2001) propose a logarithmic relationship for radiative forcing as a function CO_2, also given in IPCC (1990, chapter 2), where original sources are cited), among two other non-logarithmic, but generally concave parametrizations. IPCC (1990, chapter 2, page 51) note that for "low/moderate/high concentrations, the form is well approximated by a linear/square-root/logarithmic dependence", where the limit of validity of the logarithmic calibration is said to be 1000 ppm. For other greenhouse gases alternative parametrizations are proposed: a square-root dependence for methane and a linear dependence for halocarbons.

[13]Whereas the normalized curvature of Arrhenius's (1896) logarithmic radiative law with respect to the atmospheric carbon stock S, namely $ST_{SS}(S)/T_S(S)$ is constant and equal to -1, this limit is only reached for large carbon stock in our case, in which $\theta_E \equiv ET_{EE}(E,\chi)/T_E(E,\chi)$.

[14]The black crosses are digitized from the white crosses in Figure 2 of Allen et al. (2009) corresponding to their best fit.

Figure D.3: Temperature-carbon stock relationship for the TCRE.

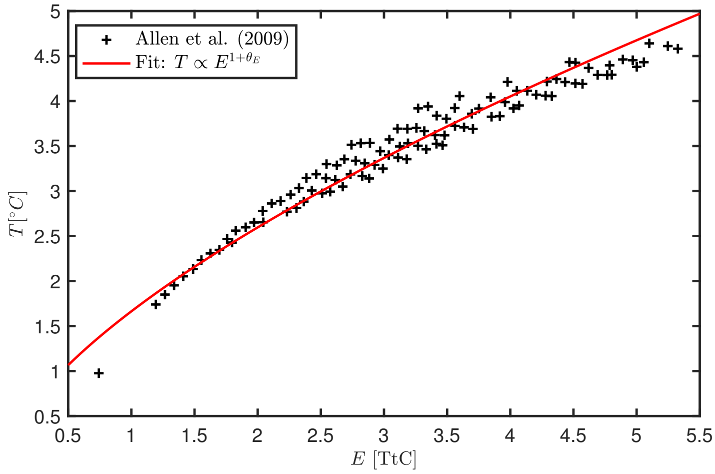


Table D.2: Climate damage calibration.

	Concave damages	**Proportional damages**	**Convex damages**
θ_T	0.50	1.0	1.5
θ_{ET}	-0.25	0	0.25
μ_λ	3.1×10^{-3}	2.2×10^{-3}	1.6×10^{-3}
Σ_λ	2.6×10^{-3}	1.6×10^{-3}	1.0×10^{-3}
$\Sigma_\lambda/\mu_\lambda$	0.83	0.76	0.66

temperature-carbon stock relationship to be $\theta_E = -0.45$. We set $\theta_E = -0.50$ for our base case calibration.

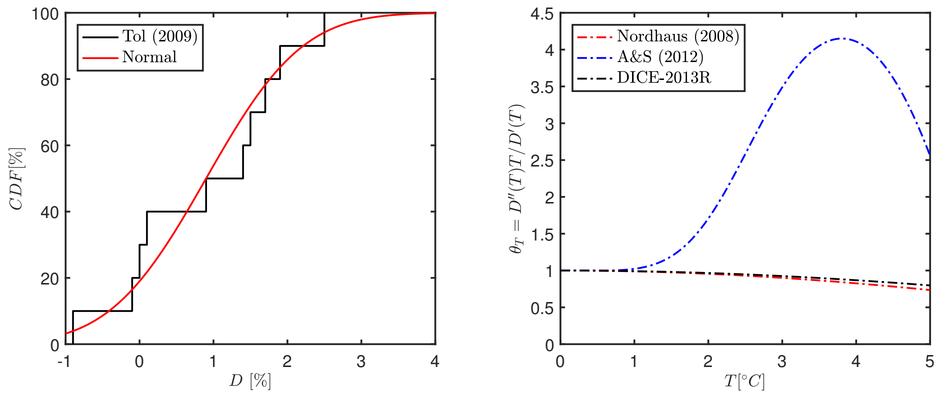
D.6.5 Climate damages and climate damage uncertainty

The empirical distribution function corresponding to the 10 damages estimates at $T = 2.5$°C reported by Tol (2009) is plotted in Figure D.4. We use the 14 damages estimates reported by Tol (2009) to estimate the mean μ_λ and the 10 damages estimates for $T = 2.5$°C to estimate the standard deviation Σ_λ for three values of $\theta_T = 1$ namely 0.5, 1 and 1.5 and report the results in Figure D.5 and Table D.2.

With $\theta_E = -0.5$, we thus have concave damages ($\theta_T = 0.5$, $\theta_{ET} = -0.25$), proportional damages ($\theta_T = 1$, $\theta_{ET} = 0$) and convex damages ($\theta_T = 1.5$, $\theta_{ET} = 0.25$), emphasizing their dependence on the carbon stock. Figure D.4b also shows the curvature $\theta_T \equiv TD''(T)/D'(T)$ of three commonly used damage specifications by Nordhaus (2008), Ackerman and Stanton (2012) and Weitzman (2012) (A&S

Figure D.4: Aspects of the climate damage calibration.

(a) Cumulative density at $T = 2.5°C$ (b) Curvature



(2012), and by DICE2013R, the last also based on the survey by Tol (2009):[15]

$$D(T) = \begin{cases} 1 - 1/\left(1 + 0.00284T^2\right) & \text{Nordhaus (2008),} \\ 1 - 1/\left(1 + 0.00245T^2 + 5.021 \times 10^{-6}T^{6.76}\right) & \text{A\&S (2012),} \\ 1 - 1/\left(1 + 0.002131T^2\right) & \text{DICE2013R.} \end{cases} \tag{D.6.2}$$

The grey arrows in Figure D.5 correspond to the ranges of damage estimates of Dietz and Stern (2008): 0.5-2% of GDP for $T = 3°C$, 1-5% for $T = 4°C$, and 1-8% for $T = 5°C$ (also used by Pindyck 2012).[16] Finally, we discuss the implications for the flow damage coefficient

$$\Theta \equiv D_E(E, \chi, \lambda) / (1 - D(E, \chi, \lambda)), \tag{D.6.3}$$

the constant of proportionality between the optimal carbon price and GDP in Results 2 and 3. Figure D.6 shows its value as function of the atmospheric carbon stock, with the values of χ and λ set to their mean values μ_χ and $\mu_\lambda$for the three values of θ_T considered in Figure D.5 and Table D.2. We set the values of μ_χ and θ_χ corresponding to the ECS, our base case (cf. Table 5.1).

The solid line in Figure D.6 shows that $\Theta(E) \equiv D'(E)/(1 - D)$ is approximately constant at 2.6% GDP/TtC for proportional damages ($\theta_T = 1$, $\theta_{ET} = 0$). For convex damages ($\theta_T = 1.5$, $\theta_{ET} = 0.25$) the flow damage coefficient starts at

[15]These damage functions turn from convex to concave at 10.8°C, 5.8°C and 12.5°C, respectively. Our power-law damage function has constant curvature making assessment of the effects of uncertainty more straightforward.

[16]Nordhaus and Sztorc (2013) also report a range of 1-5% of GDP at 4°C.

Figure D.5: Climate damage calibration.

(a) Concave damages (b) Proportional damages (c) Convex damages

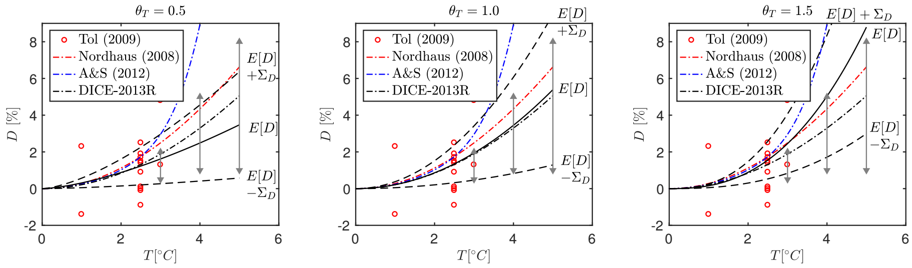
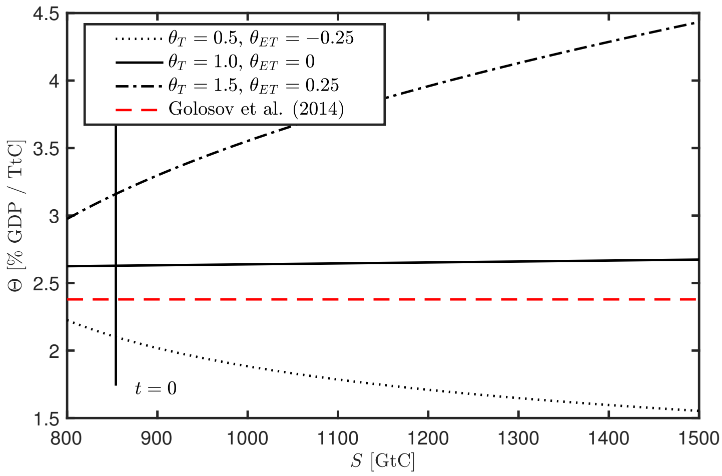


Figure D.6: Flow damage coefficient $\Theta(E)$.



3.1% GDP/TtC and then rises gradually with increasing emissions. With concave damages ($\theta_T = 0.5$, $\theta_{ET} = -0.25$), the flow damage coefficient starts at 2.1% of GDP/TtC and then gradually falls with increasing emissions. With convex (concave) damages, the optimal carbon price rises faster (slower) than GDP in our model. Golosov et al. (2014), on the other hand, have a constant flow damage coefficient of $\Theta = 3.64\%$ of GDP/TtC, which includes a markup for tipping risk.[17]

D.6.6 Climate sensitivity and uncertainty

We consider three measures of the climate sensitivity: the equilibrium climate sensitivity (ECS), the transient climate response (TCR), and the transient climate response to cumulative carbon emissions (TCRE), where the latter is defined as the change in annual mean global temperature per unit of cumulated carbon emissions

[17]Without tipping risk, $\Theta = 2.38\%$ GDP/TtC, as shown in Figure D.6.

in a scenario with continuing emissions.[18] Let χ be normally distributed with mean μ_χ and standard deviation Σ_χ.[19] The parameter θ_χ is chosen to match the potentially positively-skewed probability density functions of the climate sensitivity T_2 described by

$$f_{T_2}\left(T_2;\mu_\chi,\Sigma_\chi,\theta_\chi\right) = \frac{1}{\sqrt{2\pi}\Sigma_\chi\left(1+\theta_\chi\right)} T_2^{-\frac{\theta_\chi}{1+\theta_\chi}} \exp\left(-\left(T_2^{\frac{1}{1+\theta_\chi}} - \mu_\chi\right)^2 / 2\Sigma_\chi^2\right). \quad \text{(D.6.4)}$$

Unlike for fat-tailed distributions, which typically have algebraically-decaying tails, all moments of (D.6.4) are defined due to its exponential tail (for $\theta_\chi \geq -1$), so that Weitzman's (2009) 'dismal theorem' does not apply. Positive values of θ_χ result in a positively-skewed (non-Gaussian) distribution with more probability mass at high temperatures. Leading-order central moments of climate sensitivity can be obtained from performing Taylor-series expansions of $T_2 = \chi^{1+\theta_\chi}$ about its mean μ_χ:

$$E\left[T_2\right] = \mu_\chi^{1+\theta_\chi}\left(1 + \frac{1}{2}\theta_\chi(1+\theta_\chi)(\Sigma_\chi/\mu_\chi)^2\right) + O(\Sigma_\chi^4), \quad \text{(D.6.5a)}$$

$$\operatorname{var}\left[T_2\right] \equiv E\left[(T_2 - E\left[T_2\right])^2\right] = (1+\theta_\chi)^2\mu_\chi^{2(1+\theta_\chi)}(\Sigma_\chi/\mu_\chi)^2 + O(\Sigma_\chi^4), \quad \text{(D.6.5b)}$$

$$\operatorname{skew}\left[T_2\right] \equiv E\left[(T_2 - E\left[T_2\right])^3\right] = 3\theta_\chi(1+\theta_\chi)^3\mu_\chi^{3(1+\theta_\chi)}(\Sigma_\chi/\mu_\chi)^4 + O(\Sigma_\chi^6), \quad \text{(D.6.5c)}$$

$$\operatorname{skew}^*\left[T_2\right] \equiv \operatorname{skew}\left[T_2\right] / \left(\operatorname{var}\left[T_2\right]\right)^{3/2} = 3\theta_\chi(\Sigma_\chi/\mu_\chi) + O(\Sigma_\chi^3). \quad \text{(D.6.5d)}$$

We fit this distribution to the ECS, TCR and TCRE, respectively. Table D.3 reports the results.

Climate sensitivity and uncertainty for the ECS

To calibrate (D.6.5) for the ECS, we compare to the (thin-tailed) Gamma distribution proposed by Pindyck (2012), who considers a three-parameter Gamma distribution:[20]

$$f_{T_2,\mathrm{P}}\left(T_2; r_P, \lambda_P, \theta_P\right) = \frac{(\lambda_P)^{r_P}}{\Gamma(r_P)}\left(T_2 - \theta_P\right)^{r_P-1} e^{-\lambda_P(T_2-\theta_P)}, \quad \text{(D.6.6)}$$

[18]For low to medium estimates of climate sensitivity, the TCRE is nearly identical to the peak climate response to cumulative carbon emissions (IPCC 2013).

[19]Since the power-law transformation $T_2 = \chi^{1+\theta_\chi}$ does not allow negative values of χ, we should use a truncated normal distribution with zero probabilities for negative values of χ. In practice, these probabilities are negligibly small without truncation and we ignore this complexity, further motivated by our consideration of leading-order terms. As a result, there is a small probability atom at $T_2 = 0°$C in Figure D.4a (2.6%), which we ignore.

[20]Pindyck (2012) proposes a three-parameter gamma distribution, which allows for non-zero probability of negative temperature change. In his calibration, this probability is 2.3%. We do not allow for negative temperatures. We use the same parameter symbols as Pindyck (2012) and let subscript P denote symbols relevant to this equation only.

Table D.3: Three ways of calibrating climate sensitivity.

	ECS	**TCR**	**TCRE**
$E[T_2]$	3.0°C	1.75°C	1.9°C
$\text{var}[T_2]$	4.5°C^2	0.15°C^2	0.11°C^2
$\text{skew}[T_2]$	10°C^3	0	0.0071°C^3
$\text{skew}^*[T_2]$	1.0	0	0.19
μ_χ	1.9	1.75	1.5
Σ_χ	0.95	0.38	0.17
σ_χ	11%/year$^{1/2}$	4.5%/year$^{1/2}$	-
ν_χ	0.66%/year	$\to 0$	$\to \infty$
θ_χ	0.59	0	0.57
$\theta_{\chi T}$ ($\theta_T = 0.5$)	1.4	0.5	1.4
$\theta_{\chi T}$ ($\theta_T = 1.0$)	2.2	1.0	2.1
$\theta_{\chi T}$ ($\theta_T = 1.5$)	3.0	1.5	2.9

where $\Gamma(r_P) = \int_0^\infty s^{r_P-1}e^{-s}ds$ is the Gamma function, and the mean, variance and skewness are $r_P/ + \theta_P$, r_P/λ_P^2 and $2r_P/\lambda_P^3$, respectively. For sufficiently large temperatures, the tail in (D.6.6) always decays exponentially, so that all moments are defined. By fitting a thin-tailed Gamma distribution to a mean of 3°C, and 5% and 1% exceedance probabilities corresponding to 7°C and 10°C, respectively, Pindyck (2012) obtains a variance and skewness of 4.5 and 9.8. We fit the mean, variance and skewness of (D.6.5) to Pindyck's (2012) values and obtain $\mu_\chi = 1.9$, $\Sigma_\chi = 0.95$ and $\theta_\chi = 0.59$. Figure D.7 compares Pindyck's (2012) distribution (D.6.6) and our fitted distribution (D.6.5). The mean features of Pindyck's (2012) skewed, but thin-tailed distribution Gamma distribution, notably its high-temperature tail, are captured well by our probability distribution. It can readily be verified that (D.6.5a)-(D.6.5d) provide very accurate approximations to the first three moments.[21] Also shown in Figure D.7 is the fat-tailed distribution proposed by Roe and Baker (2007).

By setting $T_2 = \Delta T/(1-\chi)$ to reflect a standard linear feedback process, the fat-tailed distribution of Roe and Baker (2007) can be obtained by transformation of the normally distributed process χ:[22]

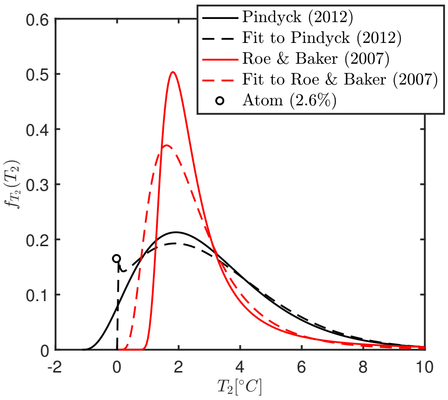
$$f_{T_2,\text{RB}}\left(T_2;\mu_{\chi,\text{RB}},\Sigma_{\chi,\text{RB}},\Delta T_{\text{RB}}\right) = \frac{1}{\sqrt{2\pi}\Sigma_{\chi,\text{RB}}}\frac{\Delta T_{\text{RB}}}{T_2^2}\exp\left(-\frac{\left(1-\mu_{\chi,\text{RB}}-\Delta T_{\text{RB}}/T_2\right)^2}{2\Sigma_{\chi,\text{RB}}^2}\right), \tag{D.6.7}$$

[21]The mean (°C), variance (°C^2) and skewness (°C^3) are, respectively: 3.0, 4.5 and 9.8 Pindyck (2012); 3.5, 4.5 and 10 (our fit) and 3.0, 4.8 and 9.7 (leading-order approximations (D.6.5a), (D.6.5b) and (D.6.5c)).

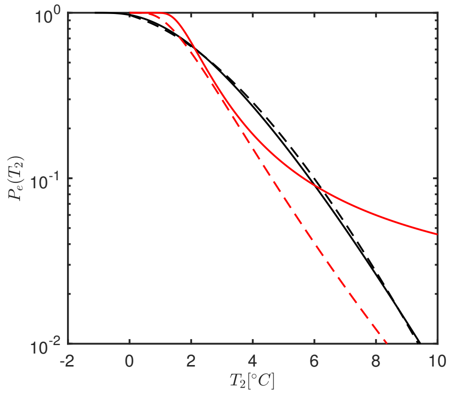
[22]Roe and Baker (2007) use f instead of χ to denote the underlying normally distributed process.

Figure D.7: Equilibrium climate sensitivity distribution.

(a) Probability density function



(b) Exceedance probability



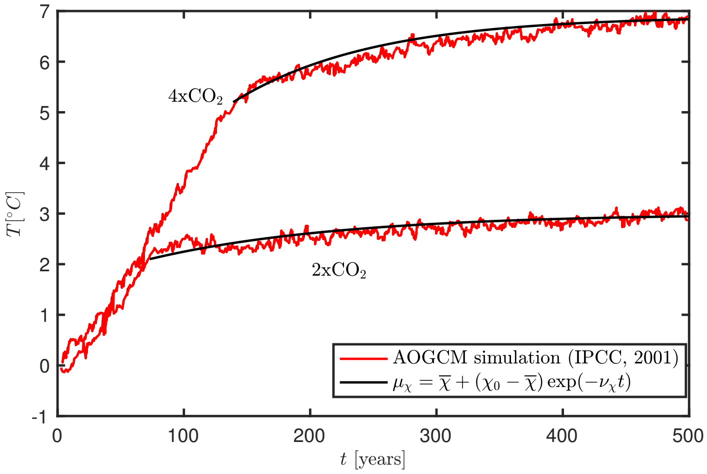
where we no longer automatically set $\Delta T = 1°$C. The slowly decaying tail causes all its moments, including mean and variance, to diverge, and the distribution can be said to be truly fat-tailed. For the Roe and Baker (2007) distribution (D.6.7), we rely on the calibration by Newbold and Daigneault (2009), who calibrate to exceedance probabilities of 1.2°C (95%), 2.4°C (50%) and 7.5°C (5%) by setting $\Delta T_{\text{RB}} = 2.04$, $\mu_{\chi,\text{RB}} = 0.14$, $\Sigma_{\chi,\text{RB}} = 0.39$. As our intention here is not to illustrate the dismal effect of fat tails, we proceed in the following completely ad-hoc fashion. To avoid unbounded moments, we truncate the probability distribution at 10°C to obtain a mean of 2.6°C, a variance of 2.1°C^2, a skewness of 6.8°C^3 and thus a standardized skewness of 2.3. We fit our distribution to these moments and obtain $\mu_P = 1.0$, $\Sigma_P = 3.9 \times 10^{-6}$ and $\theta_P = 1.5 \times 10^5$. The red dashed line in Figure D.7 illustrates that our distribution (D.6.5) cannot meaningfully reproduce the tat tails of Roe and Baker (2007). Nevertheless, both distributions are within the consensus range of IPCC (2013), who conclude that the equilibrium climate sensitivity is "extremely unlikely" (0-5%) <1°C, "likely" (66-100%) 1.5-4.5°C and "very unlikely" (0-10%) >6°C.[23]

Crucially, the equilibrium sensitivity distribution is not reached instantaneously. From the data in Figure D.8 (taken from IPCC 2001),[24] we estimate a coefficient

[23]Summing up the information presented in its Figure 10.20 and Chapter 10. The percentages in brackets correspond to the probabilities IPCC (2013) assigns to the respective measures of likelihood.

[24]We use data from the AOGCM simulation GFDL_R15_a reported in Figure 9.1 of IPCC (2001). More recent reports no longer include the simple scenario's $2xCO_2$ and $4xCO_2$, making direct estimation of ν_χ harder. The order of magnitude of our estimate of the coefficient of mean reversion agrees with more recent model runs (see IPCC 2013, Box 12.2). In reality, the response to small emissions is much faster and on a decadal scale (Ricke and Caldeira 2014) than the response to

Figure D.8: Equilibrium climate dynamics and mean reversion.



of mean reversion of $\nu_\chi = 0.66\%$/year in the scenario where CO_2 is doubled and 0.91%/year if CO_2 is quadrupled. We calibrate the ECS distribution to the large-time (or fast-mean-reversion) limit $\Sigma_\chi = \sigma_\chi/\sqrt{2\nu_\chi}$ of our Ornstein-Uhlenbeck process, relevant for the ECS (cf.

$$\Sigma_\chi = \sigma_\chi \sqrt{\left(1 - \exp\left(-2\nu_\chi t\right)\right)/2\nu_\chi} \tag{D.6.8}$$

generally). Adopting $\nu_\chi = 0.66\%$/year and using the value of Σ_χ reported in Table 5.2, the fast-mean-reversion limit gives $\sigma_\chi = 11\%/\text{year}^{1/2}$.

Climate sensitivity uncertainty based on the TCR and TCRE

To calibrate (D.6.5) for the TCR, we use information from Figure 10.20 and Chapter 10 of IPCC (2013), indicating that the TCR is "likely" (66-100%) between 1.0°C and 2.5°C and "extremely unlikely" (0-5%) to be greater than 3°C. Since there is no evidence for skewness in the TCR distribution, we set $\theta_\chi = 0$. We let the interval 1-2.5°C correspond to the 95% confidence interval of a normal distribution with mean 1.75°C, giving $\Sigma_\chi = 0.38$. Using the small-time (or slow-mean-reversion) limit $\Sigma_\chi = \sigma_\chi\sqrt{t}$, relevant for the TCR, and a period of 70 years required for doubling the atmosphere carbon stock with a rate of increase of 1.0%/year, we estimate a volatility of $\sigma_\chi = 4.5\%/\text{year}^{1/2}$.[25]

larger emissions (Zickfeld et al. 2013), reflecting non-linearity in the system, which is not captured by our Ornstein-Uhlenbeck process.

[25] With the large-time limit $\Sigma_\chi = \sigma_\chi/\sqrt{2\nu_\chi}$ and the value $\Sigma_\chi = 0.95$ for the ECS, we have an alternative way to compute mean reversion implied by the difference in volatilities of the TCR and ECS. We obtain $\nu_\chi = (0.045/0.95)^2/2 = 0.11\%$/year, which is much smaller than the value of $\nu_\chi = 0.66\%$/year.

To calibrate (D.6.2) for the TCRE, we use the 5-95% ranges reported in IPCC (2013): 0.7-2.0°C/TtC (Gillett et al. 2013), 1.0-2.1°C/TtC (Matthews et al. 2009) and 1.4-2.5°C/TtC (Allen et al. 2009). Using the values in Allen et al. (2009), who also report a best-guess of 1.9°C/TtC, we find a slightly positively-skewed distribution with $\mu_\chi = 1.5$, $\Sigma_\chi = 0.17$ and $\theta_\chi = 0.57$.[26] To reflect the transient nature of the response, we set $\nu_\chi \to \infty$, so that the large-time limit, in which volatility reaches a steady state, is instantaneously reached. To apply our model to the TCRE, one would set $\mu = 1$ and $\varphi = 0$, so that emissions stay in the atmosphere forever.[27]

D.7 Accuracy of Results 2 and 2′

Result 1 is evaluated numerically by discretization in time before evaluating the expectation operator numerically exactly and summing up the discounted contributions of every time step. Whereas the stochastic processes for χ and λ are autonomous, the stochastic process for K remains autonomous in Result 1, and all three have (independent) probability distributions available in closed form, the probability distribution of E at any time period in the future must combine all uncertain emissions (proportional to K) before that time. As the time integral of a geometric Brownian motion does not have a closed-form solution, we update the probability distribution function of E every time step with the stochastic emissions and the decay in that period according to the differential equation for E and project on a fixed grid for E to enable transfer of the probability density function between time periods. Of course, the validity of Result 1 itself still relies on the parameter ϵ being small. Consistent with our perturbation scheme, all our optimal risk-adjusted carbon prices in Results 1 and 2 or 2′ are evaluated along the business-as-usual path for which $P = 0$. We assess the accuracy of Result 2′ (or its special case Result 2) for our base line calibration and for the Gollier calibration ($\eta = \gamma = 2, \rho = 0\%$/year and $\sigma_K = 1.5\%/\text{year}^{1/2}$), as its lower discount rate $r^\star$ compared to our base case calibration makes for a more demanding test on the accuracy of Result 2′. Three factors determine the accuracy of Result 2′, as discussed below.

First, the small error due to discretization (in the states and t) associated with the evaluation of Result 1 on a grid with step size $\Delta E = 0.5$ GtC and $\Delta t = 1$ year can be estimated from grid convergence of Result 1. Second, we ignore any uncertainty in the atmospheric carbon stock that arises because of the uncertain nature of future economic growth and thus of future emissions. For our base case cali-

[26] We set the best guess of 1.9°C/TtC to equal the temperature that is most likely ($\partial f_{T_2}/\partial T_2 = 0$) and use the 5% and 95% exceedance probabilities to fit our distribution (D.6.4). We thus set $\Delta T = (1/(0.596))^{-(1+\theta_E)} = 0.75$°C with $\theta_E = -0.45$ and calibrate at $E = 1$ GtC, noting the non-linear dependence of T on E that is retained.

[27] The TCRE depends on the TCR and the cumulative airborne fraction (CAF), defined as the ratio of the increased mass of CO_2 in the atmosphere to cumulative CO_2 emissions. The CAF is equivalent to our μ. Zickfeld et al. (2013) estimate values of the CAF to the time of CO_2 doubling of 0.4-0.7, which are in line with some of the values in Table D.1 for shorter time periods.

Table D.4: Accuracy of Result 2 and 2′ compared to Result 1.

	Base case calibration		**Gollier calibration**	
	Proportional damages	Convex damages	Proportional damages	Convex damages
Error due to discretization	0.3%	1.0%	0.3%	2.6%
Error due to deterministic carbon stock	0.0%	0.0%	0%	0.1%
Error due to truncation of climate sensitivity distribution	0.1%	1.0%	0.2%	2.3%

bration with flat damages ($\theta_{ET} = 0$), the stochastic nature of E does not lead to a change in the net present value of expected damages and thus the carbon price is unchanged. For convex damages ($\theta_{ET} > 0$), the effect remains negligibly small. Third, in Result 2′ we only consider leading-order terms in the climate sensitivity uncertainty. To obtain an upper limit to the error associated with this, we numerically evaluate the expectation of $\chi^{1+\theta_{\chi T}}$ in the steady-state limit of the equilibrium sensitivity calibration and compare to the leading-order approximation used in Result 2′, $E[\chi^{1+\theta_{\chi T}}] = \mu_\chi^{1+\theta_{\chi T}}(1 + \theta_{\chi T}(1 + \theta_{\chi T})(\Sigma_\chi/\mu_\chi)^2)$. We obtain a relative error of -0.8% and -7.2% for $\theta_{ET} = 0$ (proportional damages) and $\theta_{ET} = 0.25$ (convex damages), respectively. Due to discounting over the horizon over which the climate sensitivity reaches its steady state, only part of this error manifests itself in the final estimate of the SSC, as is evident from Table D.4.